Numerical Investigation of Texturing and Wall Slip in Lubricated Sliding Contact Considering Cavitation

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(Manuscript received 05 December 2016; accepted 02 June 2017; published 30 June 2017)

In lubricated systems with textured surfaces cavitation is of importance, therefore, in the present paper, a modified Reynolds equation including wall slip is combined with a cavitation model. A numerical investigation was carried out to study the interaction of wall slip and surface texturing on the lubrication of lubricated MEMS devices. Textured surfaces with different patterns were studied and compared with the results of similar untextured surfaces. The results showed that the presence of wall slip could result in a significant reduction in the coefficient of friction. It is found that slip can improve the load support because the cavitation zone is reduced. This effect was particularly large for untextured surfaces with an optimized slip surface.

Keywords: cavitation, MEMS (micro-electro-mechanical-system), texture, wall slip

1. Introduction

Today, the performance requirements of micro-electro-mechanical-systems (MEMS) containing moving surfaces are increasingly demanding. These devices are lubricated to reduce friction and improve their life span. This means that the design of MEMS devices must undergo an optimization process for improved performance from a tribological point of view. It is generally accepted that a low friction in combination with a high load carrying capacity in lubricated MEMS is desired [1].

A great challenge in the development of MEMS lubrication is the problem of obtaining optimal tribological performance of their moving parts. This is because the lubricant behavior is different on micro- or nano-scale compared to macro-scale. On micro- or nano-scopic scale, the boundary condition will play a significant role in determining the lubricant flow behavior between the interacting components. Control of the boundary condition will allow a degree of verification over the hydrodynamic pressure in confined systems and is important in lubricated-MEMS.

Lately, studies have proven that surface texturing can be beneficial, experimentally and/or theoretically [2,3]. Its main influence is to enhance hydrodynamic pressure locally, and it results in a higher load carrying capacity of the lubricated contact.

Dobrica and Fillon [2] demonstrated that both the texture aspect ratio and Reynolds number had an equally significant effect on the validity of the Reynolds equation in analyzing a textured contact. The authors also stated that a clear loss in the overall hydrodynamic lift is partially affected by the lubricant inertia for the textured sliding contact. More investigations on the interaction of fluid inertia with surface texture in a plain sliding contact was shown by Cupillard et al. [3]. Their results indicated that inertia effects can produce either negative or positive influence on the load carrying capacity of a textured surface. A critical texture depth exists, below which positive inertia effects are encountered, and above which negative effects are observed. In relation to the friction reduction, recently, Yuan et al. [4] experimentally showed that the orientation of a micro-texture has a strong effect on the friction performance of a sliding contact. Experimentally, Yagi et al. [5] found that the formation of micro-texture on the surface of paper-friction materials can improve the dynamic friction coefficient by changing the groove volume ratio and without changing the pore characteristics of the paper. Numerically, Yagi et al. [6] investigated the magnitude of the load-carrying capacity enabled by textured feature in hydrodynamic lubrication based on the Reynolds equation. Two boundary conditions at the inlet and outlet...
were investigated and compared: an atmospheric boundary condition and a periodic boundary condition. The authors revealed that the load-carrying capacity for the atmospheric boundary condition is negative while that for the periodic boundary condition is positive when a dimple is located at the center of the lubricated area. They also found that the atmospheric boundary condition, when a dimple is located on the inlet side and is open to the outside, exhibits larger load-carrying capacity than that in the enclosed dimple case.

In several of the above-mentioned studies, an assumption of an ideal boundary condition was constructed, i.e. no boundary slip occurs at the interface. With the aim of explaining the wall slip of lubricated contacts, several lubrication theories with slip were developed [7-10]. Spikes [7] extended the Reynolds equation including boundary slip. Salant and Fortier [8] studied the slip surface with a recess without considering cavitation. Wu et al. [9], using the finite element method, proposed a modified Reynolds equation to determine the optimal slip zone. Tauviqrirrahman et al. [10] explored by an optimization approach the optimal slip zone, but for untextured slip zone. Only a few papers [11-13] refer to the combination effect of texturing and the slip on the cavitation model. Tauviqrirrahman et al. [13] investigated the combination effect of texturing and the slip on the cavitation.

\[
\frac{\partial}{\partial x} \left( \frac{h^3}{12\mu} \frac{\partial h}{\partial x} h(h + \mu \alpha) \right) + \frac{\partial}{\partial y} \left( \frac{\partial D h^2 + 4\mu \alpha}{12\mu} \frac{\partial h}{\partial y} h(h + \mu \alpha) \right) F = \frac{U}{2} \frac{\partial}{\partial x} \left( \frac{h^2 + 2h\alpha \mu}{h + \mu \alpha} \right) - \frac{U}{2} \frac{\partial}{\partial x} \left( \frac{D h^2 + 2h\alpha \mu}{h} \right) - \frac{D}{h} \frac{\partial h}{\partial x} (1 - F)
\]

It should be noted that Eq. 1 is the modified Reynolds equation derived for over the whole bearing. Universal variable \( D \) and the index \( F \) are defined to anticipate the non-active (cavitation) zone, as well as the active zone.

In the active zones: \[
\begin{align*}
D &= p, \\
F &= 1 \\
D &\geq 0
\end{align*}
\]

In the cavitation zones: \[
\begin{align*}
D &= r - h, \\
D &= 0 \\
F &= 0 \\
D &< 0
\end{align*}
\]

In this study, dealing with the derivation of modified Reynolds equation the conditions for cavitation are based on the JFO (Jakobsson-Floberg-Olsson) theory to model the rupture zone for the textured surface. The methodology used with respect to the cavitation phenomena is similar to the work of Gherca et al. [14], and is explained in detail in the Appendix.

It can be seen that if the slip length \( \alpha \) is set to zero (no-slip condition), Eq. 1 reduces to the classical Reynolds equation. In the present study, the slip length is normalized by the land film thickness \( h_F \), see Fig. 1, resulting in the dimensionless slip length \( A \). In the load support. Unfortunately, most of these references, the mass conserving cavitation which may present is neglected.

This paper presents results of the modified Reynolds equation for parallel textured lubricated sliding contacts combined with boundary slip. The aim of the analysis is to study the impact of surface texturing and slip parameters on the hydrodynamic performance with respect to friction force reduction and load carrying capacity considering cavitation.

2. Analysis

2.1. Modified Reynolds equation

The hydrodynamic performance of a textured contact with slip is investigated through numerical simulations. In order to analyze the boundary slip, the slip is modelled using the critical shear stress criterion [7]. The flow is assumed to be isothermal, isoviscous, and laminar. Figure 1 shows the schematic of a lubricated textured sliding contact in MEMS. The global contact geometry in this study is that of a rectangular textured sliding surface. It is assumed that the slip boundaries are engineered to all faces of the texture cells. In Fig. 1, an example of a texture cell with the grooves placed at the leading edge of the contact is presented. The lower body slides at a constant velocity \( U \), while the upper body is stationary. Such a lubrication system can be described by solving the three-dimensional modified Reynolds equation:

\[
\frac{\partial}{\partial x} \left( \frac{h^3}{12\mu} \frac{\partial h}{\partial x} h(h + \mu \alpha) \right) + \frac{\partial}{\partial y} \left( \frac{\partial D h^2 + 4\mu \alpha}{12\mu} \frac{\partial h}{\partial y} h(h + \mu \alpha) \right) F = \frac{U}{2} \frac{\partial}{\partial x} \left( \frac{h^2 + 2h\alpha \mu}{h + \mu \alpha} \right) - \frac{U}{2} \frac{\partial}{\partial x} \left( \frac{D h^2 + 2h\alpha \mu}{h} \right) - \frac{D}{h} \frac{\partial h}{\partial x} (1 - F)
\]

following simulations, slip coefficient \( \alpha \) is 0.02 m²/s/kg (the corresponding slip length is \( 1 \times 10^{-4} \) m) based on the results published in the literature [15,16].

2.2. Geometry and boundary conditions

The main texture parameters are the dimensionless texturing zone \( T_z \) and the texture cell aspect ratio \( \lambda \), while the slip parameter is the slip length \( \beta \), which is defined as an extrapolated distance relative to the wall where the tangential velocity component vanishes. Other parameters are the relative texture depth \( H_D \) and texture density \( T \). However, for all following computations, the parameter \( T \) and \( T_z \) is fixed at 0.5 and 0.7, respectively. The characteristics of the lubricated sliding contact are given in table 1.

2.3. Numerical method and solution

Equation 1 was solved numerically using the finite volume method. The system of equations was solved using the tridiagonal matrix algorithm (TDMA) [17]. A mesh number of about 8,000-15,000 nodes for the
computational domain was utilized depending on the texture number. It should be noted that the meshing process for textured surfaces has been checked to ensure grid independent results. In the present work, the Navier-slip model is adopted [18].

3. Results and discussion

3.1. Significance of the cavitation model

In this section, the implementation of the cavitation model in the lubrication model is discussed, i.e., the results of slider bearing with and without cavitation are compared. Figure 2 depicts a lubricated textured contact in which cavitation may occur. In this section, the positive effect of the slip will be shown as well.

Figures 3 to 4 show the calculated hydrodynamic pressures for two situations: (1) without slip but with the JFO cavitation model, (2) with boundary slip and the JFO cavitation model. Based on Fig. 3, when the JFO cavitation model is considered, the hydrodynamic pressure becomes more realistic based on the mass conservation. It is interesting to note, that when boundary slip is introduced as shown in Fig. 4, the cavitation zone disappears and this area produces more hydrodynamic pressure. It means that the slip has a significant effect on

Table 1 Characteristics of main bearing analyzed

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Symbols</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Film thickness</td>
<td>$h_F$</td>
<td>$10^{-6}$ m</td>
</tr>
<tr>
<td>Contact length in x- direction</td>
<td>$L_x$</td>
<td>$10^{-3}$ m</td>
</tr>
<tr>
<td>Contact length in y- direction</td>
<td>$L_y$</td>
<td>$10^{-3}$ m</td>
</tr>
<tr>
<td>Velocity</td>
<td>$U$</td>
<td>1 m/s</td>
</tr>
<tr>
<td>Viscosity</td>
<td>$\mu$</td>
<td>$10^{-3}$ Pa·s</td>
</tr>
</tbody>
</table>
3.2. Effect of groove depth

In the present study, it is convenient to present the simulation results with the dimensionless parameters (see Nomenclature). The first investigation deals with the effect of groove depth. As known, parallel sliding surfaces, for the classical lubricated situation i.e. no-slip condition, are not able to produce any load carrying capacity. Therefore, the groove depth is of interest.

Figure 5 shows the effect of the dimensionless groove depth $H_D$ on the dimensionless load carrying capacity $W$ and varying the number of grooves $N$. It can be seen in Fig. 5(a) that for parallel sliding surfaces, when there is no texturing (i.e. $H_D = 0$), the load carrying capacity $W$ is zero. This result is different in the case of combined texturing and slip in which a high load carrying capacity

![Image](https://via.placeholder.com/150)

Fig. 3 Three-dimensional hydrodynamic pressure distribution for a textured bearing with an inlet that starts with no-dam. No-slip and with JFO cavitation model. $H_D = 0.5$ and $N = 2$

![Image](https://via.placeholder.com/150)

Fig. 4 Three-dimensional hydrodynamic pressure distribution for a textured bearing with an inlet that starts with no-dam. Boundary slip is present and the JFO cavitation model is applied and omitted. $H_D = 0.5$ and $N = 2$

pressure generation and as a result load carrying capacity. For the numerical results in Figs. 3 and 4, the main parameters considered in the calculations are as follows: slip-texturing zone $T_R = 7 \times 10^{-4}$ m, the slip coefficient $\alpha = 0.02$ m$^2$/s/kg, pocket length $L_D = 1.4 \times 10^{-4}$ m, the pocket depth $h_D = 5 \times 10^{-7}$ m, the land film thickness $h_F = 1 \times 10^{-6}$ m, as shown in Fig. 2. It should be noted, the detail algorithm for solving the modified Reynolds equation with JFO model can be seen in Appendix.

As explained in [2] to minimize the effect of cavitation pressure, for a textured bearing the inlet should start with a dam, whilst for a bearing with boundary slip, the slip should be present in the leading edge of the contact as explained in [10]. In this paper attention is paid to bearings with optimal slip.

![Image](https://via.placeholder.com/150)

Fig. 5 Effect of the dimensionless groove depth $H_D$ on the dimensionless load carrying capacity $W$ varying the number of grooves $N$: (a) solely textured surface. (Note: $T_R/L_x = 0.7$ and $\rho = 0.5$), (b) combined textured slip surface, note the difference in scales for $W$. (Note: $T_R/L_x = 0.7$ and $\rho = 0.5$)
is generated. This is because, eventhough there are no grooves in the inlet of the lubricated contact (i.e. smooth surface), the slip at a certain area of the stationary surface with an optimized pattern is able to create hydrodynamic pressure (and thus load carrying capacity), even in the absence of the wedge (see Fig. 5(b)).

Based on Fig. 5, one can also say that, in the case of a solely textured surface, increasing the groove depth will increase the load carrying capacity. For a groove number \( N \) which is lower than 10, it is interesting to note that the increase in \( W \) is limited. There is an optimum value for the groove depth after exceeding this value, the \( W \) will decrease but not significantly. For a higher groove number (i.e. \( N > 14 \)), such a trend cannot be identified, i.e. an optimum value of \( H_D \) is not noted.

In relation to the effect of boundary slip, Fig. 5 also shows that for the case of combined slip and texturing, there is an interesting result. Boundary slip has proven that it is able to generate more hydrodynamic pressure (Fig. 6) and thus load carrying capacity, even with uniform film thickness. For all groove numbers, \( W \) is approximately 1.05 when \( H_D = 0 \). By increasing the groove depth, \( W \) increases and decreases after \( H_D = 0.2 \). From Fig. 5, the increase in \( W \) is not very significant (< 1%). After exceeding the optimum value of \( H_D \), the \( W \) decreases. This is to say, that in an untextured surface (smooth condition), as long as the surface is engineered based on an optimized slip surface, the \( W \) is high and this is a desired situation.

In addition to the load carrying capacity, the coefficient of friction (COF) is investigated. This is because, in MEMS, by lubrication, a minimum coefficient of friction means that the viscous friction force is low and is very beneficial with respect to future high-speed sliding MEMS. In this study, the coefficient of friction (COF) is determined by normalizing the friction force \( f \) to the load carrying capacity \( w \).

Figure 7 shows the effect of the groove depth \( H_D \) on the coefficient of friction for two values of the groove number \( N \). It is noted that the profiles presented in Fig. 7 are selected for a high and a low groove number. It is shown that adding boundary slip will decrease the COF significantly compared to the solely textured pattern. Compared to the case of pure texturing, the COF can decrease by a factor 20. Based on Fig. 7, concerning the effect of the dimensionless groove depth, there is a different trend of the COF for the case of pure texturing and for the combined texturing pattern with slip. For the former, the increase in groove depth has a positive effect on reducing the COF, while for the latter, increasing \( H_D \) will increase the COF especially for the case of low \( N \).

For the case of solely textured configuration, the rapid decrease in COF occurs when the groove depth is increased up to 0.5. For \( H_D \) which is larger than this

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Fig. 6 Three-dimensional hydrodynamic pressure for solely textured pattern (without slip) with \( H_D = 0.5 \) and \( N = 2 \)

Fig. 7 Effect of the dimensionless groove depth \( H_D \) on the dimensionless coefficient of friction COF for two groove numbers \( N \) in the case: (a) solely textured surface; note the difference in scales for the COF, (b) combined textured slip surface; note the difference in scales for the COF.
value, the COF is nearly insensitive to the groove depth. This trend prevails for a low as well as a high groove number. A contradictory result is found when wall slip is combined with the textured surfaces. It can be seen in Fig. 7 that, for all values of $H_D$ and $N$, the prediction of the COF is higher than that without texturing ($H_D = 0$).

Figure 8 shows the effect of the dimensionless groove depth $H_D$ and groove number $N$ on the dimensionless friction force $F_f$ for the slip and no-slip situation. The dimensionless friction force shown in Fig. 8 is nearly constant for both cases (slip and no-slip texturing). An interesting aspect to note, with respect to the friction force, is that the dimensionless friction force is independent of the textured groove number both in the case of a solely textured pattern and the combined textured slip configuration. This result matches well with the one-dimensional analytical work of Pascovisi et al. [19] from a pure texturing point of view. As mentioned earlier, it indicates that an untextured (smooth surface) with slip is recommended to be applied in real engineering applications with respect to the coefficient of friction.

3.3. Effect of groove number

Figure 9 shows the dimensionless load carrying capacity $W$ as a function of the groove number $N$ for several values of dimensionless groove depth $H_D$. It can be seen that for both cases (i.e. solely textured pattern and combined textured slip configuration), the trend of $W$ is slightly similar. Increasing the groove number decreases $W$ initially and then increases for all values of groove depth. This results in the existence of a minimum value for $W$ for a certain $H_D$. The inflection of this point differs with the groove number. For example, for pure texturing, using a high groove number, the inflection point occurs when $N$ is approximately 6, but for $H_D$ which is smaller than 0.4, there is a shift of the inflection point towards the right-hand side of the graph. It indicates that for a surface which is “rough”, it is necessary to increase the groove number in order to improve the load carrying capacity. However, it does not prevail for a smooth surface (i.e. small $H_D$). From Fig. 9, it is also observed that the inflection of $W$ in the case of the combined textured slip pattern is relatively small. The predicted results tend to be linear, especially for a low groove number. For high $H_D$, it is clearly visible the increase in $W$ after $N$ reaches the inflection point. It is interesting to note that this result is similar to the case of pure texturing. The difference lies in the inflection point,

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Fig. 8 Effect of the dimensionless groove depth $H_D$ and groove number $N$ on the dimensionless friction force $F_f$ for the slip and no-slip situation

Fig. 9 Effect of the groove number $N$ on the dimensionless load carrying capacity $W$ for several values of dimensionless groove depth $H_D$ for: (a) textured surface. (Note: insert of Fig. 9(a) the effect of the groove number $N$ for the range of 0 to 2 on the dimensionless load carrying capacity $W$), (b) combined textured slip surface. (Note: insert of Fig. 10 (b) the effect of the groove number $N$ for the range of 0 to 2 on the dimensionless load carrying capacity $W$)
which refers to a smaller groove number. As a concluding remark, for the textured pattern both with and without wall slip, increasing the groove number will be beneficial to improving the load carrying capacity. It is worth mentioning that when \( N = 1 \), the bearing configuration is similar to the Rayleigh step bearing.

4. Conclusions

In the present study, the effect of both boundary slip and surface texturing in a lubricated parallel sliding contact were analyzed. A modified Reynolds equation, that takes slip and cavitation into account, was presented and solved. The results obtained can be summarized as follows.

1. The presence of wall slip results in a reduction of the cavitation zone.
2. Under parallel-gap conditions, an effective load carrying capacity of the lubricant film can be obtained by either a solely textured surface (without slip) or a combined textured surface with boundary slip configuration.
3. In terms of the friction coefficient, an untextured slip surface is beneficial with respect to the hydrodynamic performance. In addition, slip has a positive effect to improve the load support.

Nomenclature

\[ A \quad \text{dimensionless slip coefficient} \]
\[ D \quad \text{universal function} \]
\[ COF \quad \text{dimensionless coefficient of friction} \]
\[ f \quad \text{friction force} \]
\[ F \quad \text{Switch function} \]
\[ F_f \quad \text{dimensionless friction force} \]
\[ h \quad \text{film thickness} \]
\[ h_D \quad \text{groove depth} \]
\[ H_D \quad \text{dimensionless film thickness} \]
\[ h_F \quad \text{land film thickness} \]
\[ h_i \quad \text{inlet film thickness} \]
\[ L_D \quad \text{groove length} \]
\[ L_C \quad \text{texture cell length} \]
\[ L_x \quad \text{length of lubricated surface in the } x\text{-direction} \]
\[ L_y \quad \text{length of lubricated surface in the } y\text{-direction} \]
\[ p \quad \text{fluid film pressure} \]
\[ P \quad \text{dimensionless pressure} \]
\[ r \quad \text{“filling” factor} \]
\[ T \quad \text{texture density} \]
\[ T_R \quad \text{texturing zone} \]
\[ T_Z \quad \text{dimensionless texturing zone} \]
\[ U \quad \text{velocity} \]
\[ w \quad \text{load carrying capacity}, \quad \int_0^{L_x} \int_0^{L_y} p(x,y)dx\,dy \]
\[ W \quad \text{dimensionless Load support} \]
\[ x \quad \text{coordinate in direction of sliding} \]
\[ y \quad \text{coordinate through thickness of film} \]
\[ z \quad \text{coordinate transverse to main direction of sliding} \]
\[ \alpha \quad \text{slip coefficient} \]
\[ \alpha_h \quad \text{slip coefficient at surface } h \text{ (stationary surface)} \]
\[ \beta \quad \text{slip length, } \alpha \mu \]
\[ \lambda \quad \text{texture aspect ratio} \]
\[ \mu \quad \text{dynamic viscosity} \]
\[ \rho \quad \text{density of a mixture of gas and fluid} \]
\[ \rho_o \quad \text{lubricant density} \]

Nomenclature for dimensionless values

\[ A = \frac{h}{h_F} \]
\[ H_o = \frac{h_o}{h_F} \]
\[ F_f = \frac{F_f}{(U \mu L)} \]
\[ L'_x = \frac{x}{L_x} \]
\[ L'_y = \frac{y}{L_y} \]
\[ P = \frac{p h_i^2}{(U \mu L)} \]
\[ W = \frac{w h_i^2}{(U \mu L)} \]
\[ COF = \frac{f}{w} \]
\[ \lambda = \frac{L_D}{h_D} \]
\[ T = \frac{L_D}{L_C} \]
\[ T_Z = \frac{T_R}{L_x} \]

References


Appendix

The modified Reynolds equation (Eq. 1) is derived for an incompressible lubricant with constant viscosity from a simple form of the Navier-Stokes equation neglecting the inertia in the film.

\[
\begin{align*}
\frac{\partial p}{\partial x} &= \frac{\partial}{\partial x} \left( \frac{\mu}{\partial z} \frac{\partial u}{\partial z} \right) \\
\frac{\partial p}{\partial y} &= \frac{\partial}{\partial y} \left( \frac{\mu}{\partial z} \frac{\partial v}{\partial z} \right) \\
\frac{\partial p}{\partial z} &= 0
\end{align*}
\]  

(A.1)

In order to obtain the velocity distribution by integration of Eq. (1), it is necessary to define the surface boundary conditions. The surface boundary conditions are proposed as:

at \( z=0, \ u = U; \ v = 0 \)

at \( z=h, \ u = -\alpha \mu \frac{\partial u}{\partial z} \left|_{z=h} \right.; \ v = -\alpha \mu \frac{\partial v}{\partial z} \left|_{z=h} \right. \)

(A.2)

The expression of the fluid velocity in the x- direction, subject to the above boundary conditions is:

\[
u \left( \frac{1}{2 \mu} \frac{\partial p}{\partial x} \right)^2 + h \frac{\partial p}{\partial x} \left( \frac{h + 2 \alpha \mu }{2 \mu} \right) \frac{\partial}{\partial x} \left( \frac{h + \alpha \mu }{h + \mu \alpha_s} \right) + U \frac{h + \alpha \mu }{h + \mu \alpha_s} \]

(A.3)

A similar expression can be written for the fluid velocity in the y- direction. Assuming the fluid density is constant, the conservation of mass requires

\[
\int_{0}^{h} \frac{\partial u}{\partial x} \, dz + \int_{0}^{h} \frac{\partial v}{\partial y} \, dz + \int_{0}^{h} \frac{\partial v}{\partial z} \, dz = 0
\]

(A.4)

Deriving each term of above equation (Eq. A.4), the modified Reynolds equation obtained is:

\[
\begin{align*}
\frac{\partial}{\partial x} \left( \frac{h^2 + 2 \alpha \mu }{12 \mu} \right) \frac{\partial p}{\partial x} \left( \frac{h + \mu \alpha_s}{h + \alpha \mu} \right) + \frac{\partial}{\partial y} \left( \frac{h^2 + 4 \alpha \mu }{12 \mu} \right) \frac{\partial p}{\partial y} \left( \frac{h + \mu \alpha_s}{h + \alpha \mu} \right) = \frac{U}{2} \frac{\partial}{\partial x} \left( \frac{h^2 + 2 \alpha \mu }{h + \mu \alpha_s} \right) \frac{\partial}{\partial x} \left( \frac{h + \alpha \mu }{h + \mu \alpha_s} \right) + \frac{h \alpha \mu}{2 \mu} \frac{\partial}{\partial x} \left( \frac{\partial p}{\partial x} \right) + \frac{h \alpha \mu}{2 \mu} \frac{\partial}{\partial y} \left( \frac{\partial p}{\partial y} \right)
\end{align*}
\]

(A.5)

Equation A.5 can be applied to the active and no-active zone. Thus, Eq. A.5 is modified as follows:
\[
\left\{ \frac{\partial}{\partial x} \left( \frac{h^3}{12 \mu} \frac{\partial h}{\partial x} + 4h \mu \sigma_s \right) \right\} + \frac{\partial}{\partial y} \left( \frac{h^3}{12 \mu} \frac{\partial h}{\partial y} \right) F = \frac{U}{2} \frac{\partial}{\partial x} \left( \frac{h^3}{h + \mu \sigma_s} + 2\alpha \sigma_s \mu \right)
\]

\[
= U \frac{\sigma_s \mu}{h + \mu \sigma_s} \frac{\partial h}{\partial x} - \frac{U}{2} \frac{\partial}{\partial x} \left( \frac{D h^3}{h + \mu \sigma_s} + 2h \alpha \sigma_s \mu \right) \frac{\partial h}{\partial x} \left(1 - F \right) \quad (A.6)
\]

In the active zones:
\[
\{D = p, \quad F = 1, \quad D \geq 0\}
\]

\[
\frac{\partial}{\partial x} \left( \frac{h^3}{12 \mu} \frac{\partial h}{\partial x} \right) + \frac{\partial}{\partial y} \left( \frac{h^3}{12 \mu} \frac{\partial h}{\partial y} \right) = \frac{U}{2} \frac{\partial}{\partial x} \left( \frac{h^3}{h + \mu \sigma_s} + 2h \alpha \sigma_s \mu \right) - U \frac{\sigma_s \mu}{h + \mu \sigma_s} \frac{\partial h}{\partial x} + \frac{h}{2 \mu h + \mu \sigma_s} \left( \frac{\partial \rho \partial h + \partial \rho \partial h}{\partial x \partial y} \right) \quad (A.7)
\]

In the cavitation zones:
\[
U \frac{\partial}{\partial x} \left( \frac{h^3}{h + \mu \sigma_s} \right) - U \frac{\sigma_s \mu}{h + \mu \sigma_s} \frac{\partial h}{\partial x} + \frac{U}{2} \frac{\partial}{\partial x} \left( \frac{r - h h^3}{h + \mu \sigma_s} + 2h \alpha \sigma_s \mu \right) \frac{h}{h + \mu \sigma_s} \frac{\partial h}{\partial x} = 0 \quad (A.8)
\]

where: \(D = r - h = h(\bar{p} - 1)\); \(r = \bar{p} \rho / \rho_s\), \(\rho\) is the density of lubricant-gas mixture and \(\rho_s\) the density of the lubricant.

\[
U \frac{\partial}{\partial x} \left( \rho \frac{h^3}{h + \mu \sigma_s} \right) - U \frac{\sigma_s \mu}{h + \mu(\sigma_s + 1)} \frac{\partial h}{\partial x} = 0 \quad (A.9)
\]

In order to simultaneously solve Eqs. (A.5-9) over the whole domain of the bearing, a universal variable \(D\) is defined and an unified modified Reynolds equation is obtained as stated in Eq. 1, section 2. As a note, the universal function \(D\) in A.6 for the no-cavitation case (i.e. active zone) refers to pressure or \(p\). However, for the cavitation zone, the \(D\) in A.6 refers to the function of the density (lubricant and gas/vapor). So, in the cavitation zone, the vapor density can be calculated from Eq. A.9 above. Then, based on the continuity equation, the pressure in the cavitation region can be obtained. It should be highlighted that in Eq. A.9, the value \(D\) (in which \(D = r - h\)) in the cavitation region is always smaller than 1.