LAMPIRAN A
PENURUNAN REYNOLD *NON-NEWTONIAN*

A.1 Penurunan Persamaan Reynold

Persamaan momentum untuk aliran satu dimensi:

$$\frac{\partial p}{\partial x} = \frac{\partial \tau}{\partial z} \tag{A.1}$$

Model *power law* untuk fluida *non-Newtonian*

$$\tau = \eta \left|\frac{\partial u}{\partial z}\right|^{n-1} \frac{\partial u}{\partial z} = \eta_e \frac{\partial u}{\partial z} \tag{A.2}$$

dimana

$$\eta_e = \eta \left|\frac{\partial u}{\partial z}\right|^{n-1} = viskositas\,ekuivalen \tag{A.3}$$

Kecepatan u ditentukan dengan istilah ε, yang merupakan parameter amplitudo tak berdimensi yang kecil. (Dien & Elrod, 1983)

$$u = u_0(x,z) + \varepsilon u_1(x,z) + ... \tag{A.4}$$

misal $I = \frac{\partial u}{\partial z}$ (A.5)

maka $I = \frac{\partial u}{\partial z} = \frac{\partial u_0}{\partial z} + \varepsilon \frac{\partial u_1}{\partial z}$ (A.6)

dapat kita tulis menjadi $I = I_0 + \varepsilon I_1$ (A.7)

dimana $I_0 = \frac{\partial u_0}{\partial z}$ dan $I_1 = \frac{\partial u_1}{\partial z}$ (A.8)

Dengan menggunakan metode yang sama, viskositas ekuivalen, η_e dan tekanan *fluid film*, *p* dapat diuraikan dalam bentuk

$$\eta_e = \eta_0 + \varepsilon\eta_1 \quad \text{(A.9)}$$

$$p = p_0 + \varepsilon p_1 \quad \text{(A.10)}$$

Dari persamaan (A.2)

$$\begin{aligned}\tau &= \eta_e \frac{\partial u}{\partial z} \\ &= \eta_e I \\ &= \eta_e (I_0 + \varepsilon I_1) \\ &= (\eta_0 + \varepsilon\eta_1)(I_0 + \varepsilon I_1) \\ &= \eta_0 I_0 + \varepsilon(\eta_0 I_1 + \eta_1 I_0) + \varepsilon^2 \eta_1 I_1 \quad \text{(A.11)}\end{aligned}$$

dengan mengabaikan nilai ε^2 (terlalu kecil), menjadi

$$\tau = \eta_0 I_0 + \varepsilon(\eta_0 I_1 + \eta_1 I_0) + \varepsilon^2 \eta_1 I_1 \quad \text{(A.12)}$$

Substitusikan persamaan (A.10) dan persamaan (A.12) ke dalam persamaan (A.1)

$$\frac{\partial p}{\partial x} = \frac{\partial \tau}{\partial z}$$

$$\frac{\partial}{\partial x}(p_0 + \varepsilon p_1) = \frac{\partial}{\partial z}\left[\eta_0 I_0 + \varepsilon(\eta_0 I_1 + \eta_1 I_0)\right]$$

$$\frac{\partial p_0}{\partial x} + \varepsilon \frac{\partial p_1}{\partial x} = \eta_0 \frac{\partial I_0}{\partial z} + \varepsilon \frac{\partial}{\partial z}(\eta_0 I_1 + \eta_1 I_0)$$

(A.13)

Dengan asumsi aliran adalah Couette dominated, gradien tekanan dapat ditulis sebagai

$$\nabla p = \varepsilon \nabla p_1 \quad \text{(A.14)}$$

dengan $\frac{\partial p_0}{\partial x} = 0$

Jadi, dari persamaan (A.13):

$$\varepsilon \frac{\partial p_1}{\partial x} = \eta_0 \frac{\partial I_0}{\partial z} + \varepsilon \frac{\partial}{\partial z}(\eta_0 I_1 + \eta_1 I_0) \quad \text{(A.15)}$$

dengan menyamakan koefisien dari *equal power* dari ε, kita peroleh

$$\eta_0 \frac{\partial I_0}{\partial z} = 0$$

$$\eta_0 \frac{\partial^2 u_0}{\partial z^2} = 0 \tag{A.16}$$

dan

$$\frac{\partial p_1}{\partial x} = \frac{\partial}{\partial z}(\eta_0 I_1 + \eta_1 I_0)$$

$$\frac{\partial p_1}{\partial x} = \frac{\partial}{\partial z}\left(\eta_0 \frac{\partial u_1}{\partial z} + \eta_1 \frac{\partial u_0}{\partial z}\right) \tag{A.17}$$

Integralkan persamaan (A.16) dengan kondisi batas

$$u_0 = u_b \quad \text{saat} \quad z = 0$$

$$u_0 = u_s \quad \text{saat} \quad z = h$$

Jadi, dari persamaan (A.16)

$$\frac{\partial u_0}{\partial z} + c_1 = 0 \tag{A.18}$$

$$u_0 + c_1 z + c_2 = 0 \tag{A.19}$$

saat $z = 0$, $u_0 = u_b$;

$$u_b + c_1(0) + c_2 = 0$$

$$c_2 = -u_b \tag{A.20}$$

saat $z = h$, $u_0 = u_s$;

$$u_s + c_1 h - u_b = 0$$

$$c_1 = \frac{u_b - u_s}{h} \tag{A.21}$$

Dari persamaan (A.19)

$$u_0 + \left(\frac{u_b - u_s}{h} \right) z - u_b = 0$$

jadi $$u_0 = u_b + \left(\frac{u_s - u_b}{h} \right) z \qquad \text{(A.22)}$$

dimana: u_b = kecepatan permukaan bawah *(moving surface)*

u_s = kecepatan slip *(stationary surface)*

h = tebal lapisan *(film thickness)*

Dari persamaan (9), viskositas ekuivalen, η_e dapat ditulis sebagai

$$\eta_e = \eta_0 + \varepsilon \eta_1$$

dimana

$$\eta_0 = \eta_e(I_0) \qquad \text{(A.23)}$$

$$\eta_1 = \left(\frac{\partial \eta_e}{\partial I} \right)_{I=I_0} .I_1 \qquad \text{(A.24)}$$

Substitusikan persamaan (A.23) dan (A.24) kedalam (A.17)

$$\frac{\partial p_1}{\partial x} = \frac{\partial}{\partial z} \left\{ \eta_e I_0 I_1 + I_1 \left(\frac{\partial \eta_e}{\partial I} \right)_{I=I_0} I_0 \right\}$$

$$\frac{\partial p_1}{\partial x} = \frac{\partial I_1}{\partial z} \left\{ \eta_e I_0 + \left(\frac{\partial \eta_e}{\partial I} \right)_{I=I_0} I_0 \right\}$$

jadi $$\frac{\partial p_1}{\partial x} = \eta' \frac{\partial^2 u_1}{\partial z^2} \qquad \text{(A.25)}$$

dimana $$\eta' = \left[\eta_0 + I_0 \left(\frac{\partial \eta_e}{\partial I} \right)_{I=I_0} \right] \qquad \text{(A.26)}$$

Integralkan persamaan (A.25) dengan kondisi batas

$$u_1 = 0 \text{ saat } z = 0$$

$$u_1 = 0 \text{ saat } z = h$$

$$\eta' \frac{\partial^2 u_1}{\partial z^2} = \frac{\partial p_1}{\partial x}$$

$$\eta' u_1 = \frac{z^2}{2}\frac{\partial p_1}{\partial x} + c_1 z + c_2 \tag{A.27}$$

saat $z = 0$, $u_1 = 0$;

$$0 = 0 + c_1(0) + c_2$$
$$c_2 = 0$$

saat $z = h$, $u_1 = 0$;

$$0 = \frac{h^2}{2}\frac{\partial p_1}{\partial x} + c_1 h$$
$$c_1 h = -\frac{h^2}{2}\frac{\partial p_1}{\partial x}$$
$$c_1 = -\frac{h}{2}\frac{\partial p_1}{\partial x}$$

Jadi, dari persamaan (A.27)

$$\eta' u_1 = \frac{z^2}{2}\frac{\partial p_1}{\partial x} + \frac{hz}{2}\frac{\partial p_1}{\partial x}$$

$$u_1 = \frac{(z^2 - hz)}{2\eta'}\frac{\partial p_1}{\partial x} \tag{A.28}$$

Dari persamaan (A.4)

$$u = u_0 + \varepsilon u_1 \tag{A.29}$$

Substitusikan persamaan (A.22) dan (A.28) ke dalam persamaan (A.29)

$$u = \left[u_b + \left(\frac{u_s - u_b}{h} \right) z \right] + \varepsilon \left[\frac{(z^2 - hz)}{2\eta'} \frac{\partial p_1}{\partial x} \right] \qquad \text{(A.30)}$$

Dari persamaan (A.10)

$$p = p_0 + \varepsilon p_1$$

$$p_1 = \frac{1}{\varepsilon}(p - p_0)$$

$$\frac{\partial p_1}{\partial x} = \frac{1}{\varepsilon}\left(\frac{\partial p}{\partial x} - \frac{\partial p_0}{\partial x} \right)$$

Dengan asumsi aliran adalah Couette dominated

$$\frac{\partial p_1}{\partial x} = \frac{1}{\varepsilon}\left(\frac{\partial p}{\partial x} \right) \qquad \text{(A.31)}$$

Substitusikan persamaan (A.31) ke dalam persamaan (A.30)

$$u = \left[u_b + \left(\frac{u_s - u_b}{h} \right) z \right] + \left[\frac{(z^2 - hz)}{2\eta'} \frac{\partial p}{\partial x} \right] \qquad \text{(A.32)}$$

Dari persamaan (A.23) $\qquad \eta_0 = \eta_e(I_0)$

dimana dari persamaan (A.3), viskositas ekuivalan, η_e didefinisikan sebagai:

$$\eta_e(I) = \eta \left| \frac{\partial u}{\partial z} \right|^{n-1} = \eta \left| I \right|^{n-1}$$

jadi $\qquad \eta_0 = \eta \left| I_0 \right|^{n-1} \qquad$ (A.33)

dan $\qquad \left(\frac{\partial \eta_e}{\partial I} \right)_{I=I_0} = (n-1)\eta \left| I_0 \right|^{n-2}$

$$= (n-1)\eta \frac{\left| I_0 \right|^{n-1}}{I_0} \qquad \text{(A.34)}$$

Dari persamaan (A.26)

$$\eta' = \eta_0 + I_0 \left(\frac{\partial \eta_e}{\partial I} \right)_{I=I_0}$$

Substitusikan persamaan (A.33) dan (A.34) ke dalam persamaan (A.26)

$$\begin{aligned} \eta' &= \eta \left| I_0 \right|^{n-1} + I_0 \left[(n-1)\eta \frac{\left| I_0 \right|^{n-1}}{I_0} \right] \\ &= \eta \left| I_0 \right|^{n-1} + n\eta \left| I_0 \right|^{n-1} - \eta \left| I_0 \right|^{n-1} \\ &= n\eta \left| I_0 \right|^{n-1} \\ &= n\eta \left| \frac{\partial u_0}{\partial z} \right|^{n-1} \end{aligned}$$

$$\eta' = n\eta \left| \frac{u_b - u_s}{h} \right|^{n-1} \qquad \text{(A.35)}$$

Substitusikan persamaan (A.35) ke dalam persamaan (A.32)

$$u = \left[u_b + \left(\frac{u_s - u_b}{h} \right) z \right] + \left[\frac{(z^2 - hz)}{2n\eta \left(\frac{u_b - u_s}{h} \right)^{n-1}} \frac{\partial p}{\partial x} \right] \qquad \text{(A.36)}$$

Persamaan kontinuitas untuk aliran fluida tiga dimensi (3D) dinyatakan sebagai

$$\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial x} + \frac{\partial \rho w}{\partial x} = 0 \qquad \text{(A.37)}$$

Untuk aliran satu dimensi arah x, persamaan (A.37) disederhanakan menjadi

$$\frac{\partial \rho u}{\partial x} = 0 \qquad \text{(A.38)}$$

Integralkan persamaan (A.38) dengan batas $z = 0$ sampai $z = h$

$$\int_0^h \frac{\partial \rho u}{\partial x} dz = 0 \qquad \text{(A.39)}$$

Substitusikan persamaan kecepatan, u dari persamaan (A.36) ke dalam persamaan (A.39)

$$\frac{\partial}{\partial x}\int_0^h \rho\left[u_b+\left(\frac{u_s-u_b}{h}\right)z+\frac{(z^2-hz)}{2n\eta\left(\frac{u_b-u_s}{h}\right)^{n-1}}\frac{\partial p}{\partial x}\right]dz=0$$

$$\frac{\partial \rho}{\partial x}\left[u_b z+\left(\frac{u_s-u_b}{2h}\right)z^2+\frac{(\frac{z^3}{3}-\frac{hz^2}{2})}{2n\eta\left(\frac{u_b-u_s}{h}\right)^{n-1}}\frac{\partial p}{\partial x}\right]_{y=0}^{y=h}=0$$

$$\frac{\partial \rho}{\partial x}\left[u_b h+\left(\frac{u_s-u_b}{2h}\right)h^2-\frac{(h^3)}{12n\eta\left(\frac{u_b-u_s}{h}\right)^{n-1}}\frac{\partial p}{\partial x}\right]=0$$

$$\frac{\partial \rho}{\partial x}\left[u_b h+\left(\frac{u_s-u_b}{2}\right)h-\frac{(h^3)}{12n\eta\left(\frac{u_b-u_s}{h}\right)^{n-1}}\frac{\partial p}{\partial x}\right]=0$$

$$\frac{\partial \rho}{\partial x}\left[u_b h+\left(\frac{u_s-u_b}{2}\right)h-\frac{h^3.h^{n-1}}{12n\eta\left(u_b-u_s\right)^{n-1}}\frac{\partial p}{\partial x}\right]=0$$

$$\frac{\partial \rho}{\partial x}\left[u_b h+\left(\frac{u_s-u_b}{2}\right)h-\frac{h^{n+2}}{12n\eta\left(u_b-u_s\right)^{n-1}}\frac{\partial p}{\partial x}\right]=0$$

$$\frac{\partial \rho}{\partial x}\frac{h^{n+2}\frac{\partial p}{\partial x}}{12n\eta\left(u_b-u_s\right)^{n-1}}=\frac{\partial \rho}{\partial x}\left[\left(\frac{u_b+u_s}{2}\right)h\right]$$

$$\frac{\partial}{\partial x}\frac{h^{n+2}}{n}\frac{\partial p}{\partial x}=12\eta\left(u_b-u_s\right)^{n-1}\left(\frac{u_b+u_s}{2}\right)\frac{\partial}{\partial x}h$$

(A.40)

A.2 Pembuktian

Jika n=1 dan u_s=0 maka persamaan (A.40) akan menjadi persamaan reynold klasik dimana $\eta = \mu^n$

$$\frac{\partial}{\partial x}\left(\frac{h^{n+2}}{n}\frac{\partial p}{\partial x}\right) = 12\mu^n \left(u_b - u_s\right)^{n-1}\left(\frac{u_b + u_s}{2}\right)\frac{\partial}{\partial x}h$$

$$\frac{\partial}{\partial x}\left(\frac{h^{1+2}}{1}\frac{\partial p}{\partial x}\right) = 12\mu^1 \cancel{\left(u_b - 0\right)^{1-1}}\left(\frac{u_b + 0}{2}\right)\frac{\partial}{\partial x}h$$

$$\frac{\partial}{\partial x}\left(h^3\frac{\partial p}{\partial x}\right) = 12\mu\left(\frac{u_b}{2}\right)\frac{\partial}{\partial x}h$$

$$\frac{\partial}{\partial x}\left(h^3\frac{\partial p}{\partial x}\right) = 6\mu u_b \frac{\partial}{\partial x}h$$