LAMPIRAN A

A.1. Penurunan Rumus Navier *Slip* 2 Dimensi – 2 *Slip*

Persamaan Reynold Isoviskos 2-D diturunkan dari bentuk sederhana persamaan Navier-Stokes yang mengasumsikan sebuah aliran laminar dengan mengabaikan efek inersia pada lapisan film:

$$\left.\begin{aligned} \frac{\partial p}{\partial x} &= \frac{\partial}{\partial z}\left(\mu \frac{\partial u}{\partial z}\right) \\ \frac{\partial p}{\partial y} &= \frac{\partial}{\partial z}\left(\mu \frac{\partial v}{\partial z}\right) \\ \frac{\partial p}{\partial z} &= 0 \end{aligned}\right\} \qquad (1)$$

Persamaan pertama harus di integrasi untuk mendapatkan kecepatan fluida. Sebelum di integrasi, dilakukan pendefinisian kondisi batas. *Slip* hanya akan terjadi pada area dimana permukaan *housing* ataupun poros (*shaft*) yang telah diperlakukan dan ketika tegangan geser melebihi tegangan geser kritis τ_c. Ketika kedua persyaratan ini dipenuhi maka dihasilkan kecepatan *slip* yang perbedaannya proporsional antara nilai tegangan geser dan tegeangan geser kritis, dengan faktor proporsionalitas α_h untuk *housing* dan α_s untuk poros. Dengan menganggap bahwa tegangan geser kritis adalah nol, maka kondisi batasnya adalah:

$$\left.\begin{aligned} &pada\, z=0,\; u = U + \alpha_s \mu \left.\frac{\partial u}{\partial z}\right|_{z=0} \qquad ;\; v = \alpha_s \mu \left.\frac{\partial v}{\partial z}\right|_{z=0} \\ &pada\, z=h,\quad u = -\alpha_h \mu \left.\frac{\partial u}{\partial z}\right|_{z=h} \qquad ;\; v = -\alpha_h \mu \left.\frac{\partial v}{\partial z}\right|_{z=h} \end{aligned}\right\} \qquad (2)$$

Penurunan rumus untuk arah x

$$\frac{\partial^2 u}{\partial z^2} = \frac{1}{\mu}\frac{\partial p}{\partial x}$$

$$u = \frac{1}{2\mu}\frac{\partial p}{\partial x}Z^2 + C_1 Z + C_2 \qquad \text{(a)}$$

Pemasukan kondisi batas

- pada $z = 0$,

$$u = U + \alpha_s \mu \left.\frac{\partial u}{\partial z}\right|_{z=0}$$

$$U + \alpha_s \mu \frac{\partial u}{\partial z} = \frac{1}{2\mu}\frac{\partial p}{\partial x}Z^2 + C_1 Z + C_2$$

$$U + \alpha_s \mu \left(\frac{1}{\mu}\frac{\partial p}{\partial x}Z + C_1 \right) = \frac{1}{2\mu}\frac{\partial p}{\partial x}Z^2 + C_1 Z + C_2$$

$$U + \alpha_s Z \frac{\partial p}{\partial x} + \alpha_s \mu C_1 = \frac{1}{2\mu}\frac{\partial p}{\partial x}Z^2 + C_1 Z + C_2$$

$$U + \alpha_s .(0)\frac{\partial p}{\partial x} + \alpha_s \mu C_1 = \frac{1}{2\mu}\frac{\partial p}{\partial x}\left(0\right)^2 + C_1(0) + C_2$$

$$U + \alpha_s \mu C_1 = C_2$$

- pada $z = h$,

$$u = -\alpha_h \mu \left.\frac{\partial u}{\partial z}\right|_{z=h}$$

$$-\alpha_h \mu \frac{\partial u}{\partial z} = \frac{1}{2\mu}\frac{\partial p}{\partial x}z^2 + C_1 Z + C_2$$

$$-\alpha_h \mu \left(\frac{1}{\mu}\frac{\partial p}{\partial x}Z + C_1 \right) = \frac{1}{2\mu}\frac{\partial p}{\partial x}Z^2 + C_1 Z + C_2$$

$$-\alpha_h Z \frac{\partial p}{\partial x} - \alpha_h \mu C_1 = \frac{1}{2\mu}\frac{\partial p}{\partial x}Z^2 + C_1 Z + C_2$$

$$-\alpha_h\left(h\right)\frac{\partial p}{\partial x}-\alpha_h\mu C_1=\frac{\left(h^2\right)}{2\mu}\frac{\partial p}{\partial x}+C_1\left(h\right)+U+\alpha_s\mu C_1$$

$$-\alpha_h\mu C_1-hC_1-\alpha_s\mu C_1=\left(\frac{h^2}{2\mu}+\alpha_h h\right)\frac{\partial p}{\partial x}+U$$

$$-\left(h+\mu\left(\alpha_h+\alpha_s\right)\right)C_1=\left(\frac{h^2}{2\mu}+\alpha_h h\right)\frac{\partial p}{\partial x}+U$$

$$-C_1=\left(\frac{h^2+2\alpha_h\mu h}{2\mu}\frac{\partial p}{\partial x}.\frac{1}{h+\mu\left(\alpha_h+\alpha_s\right)}\right)+\frac{U}{h+\mu\left(\alpha_h+\alpha_s\right)}$$

$$\therefore C_1=-\left(\frac{h}{2\mu}\frac{\partial p}{\partial x}\frac{h+2\alpha_h\mu}{h+\mu\left(\alpha_h+\alpha_s\right)}+\frac{U}{h+\mu\left(\alpha_h+\alpha_s\right)}\right) \qquad \text{(b)}$$

Mencari nilai C_2

$$C_2=U+\alpha_s\mu C_1$$

$$C_2=U+\alpha_s\mu\left[-\left(\frac{h}{2\mu}\frac{\partial p}{\partial x}\frac{h+2\alpha_h\mu}{h+\mu\left(\alpha_h+\alpha_s\right)}+\frac{U}{h+\mu\left(\alpha_h+\alpha_s\right)}\right)\right]$$

$$=U-\frac{h}{2\mu}\frac{\partial p}{\partial x}\frac{\alpha_s\mu\left(h+2\alpha_h\mu\right)}{h+\mu\left(\alpha_h+\alpha_s\right)}-\frac{U\alpha_s\mu}{h+\mu\left(\alpha_h+\alpha_s\right)}$$

$$=\frac{U\left(h+\mu\left(\alpha_h+\alpha_s\right)\right)-U\alpha_s\mu}{h+\mu\left(\alpha_h+\alpha_s\right)}-\frac{h}{2\mu}\frac{\partial p}{\partial x}\frac{\alpha_s\mu\left(h+2\alpha_h\mu\right)}{h+\mu\left(\alpha_h+\alpha_s\right)}$$

$$=\frac{Uh+U\alpha_h\mu+U\alpha_s\mu-U\alpha_s\mu}{h+\mu\left(\alpha_h+\alpha_s\right)}-\frac{h}{2\mu}\frac{\partial p}{\partial x}\frac{\alpha_s\mu\left(h+2\alpha_h\mu\right)}{h+\mu\left(\alpha_h+\alpha_s\right)}$$

$$=\frac{Uh+U\alpha_h\mu}{h+\mu\left(\alpha_h+\alpha_s\right)}-\frac{h}{2\mu}\frac{\partial p}{\partial x}\frac{\alpha_s\mu\left(h+2\alpha_h\mu\right)}{h+\mu\left(\alpha_h+\alpha_s\right)}$$

$$\therefore C_2=U\frac{h+\alpha_h\mu}{h+\mu\left(\alpha_h+\alpha_s\right)}-\frac{h}{2\mu}\frac{\partial p}{\partial x}\frac{\alpha_s\mu\left(h+2\alpha_h\mu\right)}{h+\mu\left(\alpha_h+\alpha_s\right)} \qquad \text{(c)}$$

Substitusi pers (b) dan (c) ke (a) sehingga didapat kecepatan aliran pada arah x yaitu

$$u = \frac{1}{2\mu}\frac{\partial p}{\partial x}z^2 - \left(\frac{U}{h + \mu\left(\alpha_h + \alpha_s\right)} + \frac{h}{2\mu}\frac{\partial p}{\partial x}\frac{h + 2\alpha_h\mu}{h + \mu\left(\alpha_h + \alpha_s\right)} \right) z + U\frac{h + \alpha_h\mu}{h + \mu\left(\alpha_h + \alpha_s\right)}$$

$$-\frac{h}{2\mu}\frac{\partial p}{\partial x}\frac{\alpha_s\mu(h + 2\alpha_h\mu)}{h + \mu\left(\alpha_h + \alpha_s\right)} \qquad (3)$$

Penurunan rumus untuk arah y

$$\frac{\partial^2 v}{\partial z^2} = \frac{1}{\mu}\frac{\partial p}{\partial y}$$

$$v = \frac{1}{2\mu}\frac{\partial p}{\partial y}Z^2 + C_1 Z + C_2 \qquad \text{(d)}$$

Pemasukan kondisi batas

- pada $z = 0$,

$$v = \alpha_s \mu \left.\frac{\partial v}{\partial z}\right|_{z=0}$$

$$\alpha_s \mu \frac{\partial v}{\partial z} = \frac{1}{2\mu}\frac{\partial p}{\partial y}Z^2 + C_1 Z + C_2$$

$$\alpha_s \mu \left(\frac{1}{\mu}\frac{\partial p}{\partial y}Z + C_1\right) = \frac{1}{2\mu}\frac{\partial p}{\partial y}Z^2 + C_1 Z + C_2$$

$$\alpha_s Z \frac{\partial p}{\partial y} + \alpha_s \mu C_1 = \frac{1}{2\mu}\frac{\partial p}{\partial y}Z^2 + C_1 Z + C_2$$

$$\alpha_s .(0)\frac{\partial p}{\partial y} + \alpha_s \mu C_1 = \frac{1}{2\mu}\frac{\partial p}{\partial y}(0)^2 + C_1(0) + C_2$$

$$\alpha_s \mu C_1 = C_2$$

- pada $z = h$,

$$v = -\alpha_h \mu \left.\frac{\partial v}{\partial z}\right|_{z=h}$$

$$-\alpha_h \mu \frac{\partial v}{\partial z} = \frac{1}{2\mu}\frac{\partial p}{\partial y}z^2 + C_1 Z + C_2$$

$$-\alpha_h \mu \left(\frac{1}{\mu}\frac{\partial p}{\partial y}Z + C_1\right) = \frac{1}{2\mu}\frac{\partial p}{\partial y}Z^2 + C_1 Z + C_2$$

$$-\alpha_h Z \frac{\partial p}{\partial y} - \alpha_h \mu C_1 = \frac{1}{2\mu}\frac{\partial p}{\partial y}Z^2 + C_1 Z + C_2$$

$$-\alpha_h\left(h\right)\frac{\partial p}{\partial y}-\alpha_h\mu C_1=\frac{\left(h^2\right)}{2\mu}\frac{\partial p}{\partial y}+C_1\left(h\right)+\alpha_s\mu C_1$$

$$-\alpha_h\mu C_1-hC_1-\alpha_s\mu C_1=\left(\frac{h^2}{2\mu}+\alpha_h h\right)\frac{\partial p}{\partial y}$$

$$-\left(h+\mu\left(\alpha_h+\alpha_s\right)\right)C_1=\left(\frac{h^2}{2\mu}+\alpha_h h\right)\frac{\partial p}{\partial y}$$

$$-C_1=\left(\frac{h^2+2\alpha_h\mu h}{2\mu}\frac{\partial p}{\partial y}\cdot\frac{1}{h+\mu\left(\alpha_h+\alpha_s\right)}\right)$$

$$\therefore C_1=-\left(\frac{h}{2\mu}\frac{\partial p}{\partial y}\frac{h+2\alpha_h\mu}{h+\mu\left(\alpha_h+\alpha_s\right)}\right) \qquad \text{(e)}$$

Mencari nilai C_2

$$C_2=\alpha_s\mu C_1$$

$$C_2=\alpha_s\mu\left[-\left(\frac{h}{2\mu}\frac{\partial p}{\partial y}\frac{h+2\alpha_h\mu}{h+\mu\left(\alpha_h+\alpha_s\right)}\right)\right]$$

$$\therefore C_2=-\left(\frac{h}{2\mu}\frac{\partial p}{\partial y}\frac{\alpha_s\mu\left(h+2\alpha_h\mu\right)}{h+\mu\left(\alpha_h+\alpha_s\right)}\right) \qquad \text{(f)}$$

Substitusi pers (e) dan (f) ke (d) sehingga didapat kecepatan aliran pada arah y yaitu

$$v=\frac{1}{2\mu}\frac{\partial p}{\partial y}Z^2-\left(\frac{h}{2\mu}\frac{\partial p}{\partial y}\frac{h+2\alpha_h\mu}{h+\mu\left(\alpha_h+\alpha_s\right)}\right)Z-\left(\frac{h}{2\mu}\frac{\partial p}{\partial y}\frac{\alpha_s\mu\left(h+2\alpha_h\mu\right)}{h+\mu\left(\alpha_h+\alpha_s\right)}\right) \qquad (4)$$

Jika masa jenis fluida diasumsikan konstan, maka kekekalan massa yang dibutuhkan

$$\int_0^h \frac{\partial u}{\partial x}dz + \int_0^h \frac{\partial v}{\partial y}dz + \int_0^h \frac{\partial w}{\partial z}dz = 0 \qquad (5)$$

Maka *flow rate*/laju aliran pada arah x adalah

$$q_x = \int_0^h u\,dz - u(h)\frac{dh}{dx} \qquad (6)$$

Untuk $q_x = \int_0^h u\,dz$

$$= \int_0^h \frac{1}{2\mu}\frac{\partial p}{\partial x}Z^2 - \left(\frac{h}{2\mu}\frac{\partial p}{\partial x}\frac{h+2\alpha_h\mu}{h+\mu(\alpha_h+\alpha_s)} + \frac{U}{h+\mu(\alpha_h+\alpha_s)}\right)Z + U\frac{h+\alpha_h\mu}{h+\mu(\alpha_h+\alpha_s)}$$

$$-\frac{h}{2\mu}\frac{\partial p}{\partial x}\frac{\alpha_s\mu(h+2\alpha_h\mu)}{h+\mu(\alpha_h+\alpha_s)}dz$$

$$= \frac{1}{6\mu}\frac{\partial p}{\partial x}Z^3 - \frac{U}{2}\frac{Z^2}{h+\mu(\alpha_h+\alpha_s)} - \frac{h}{4\mu}\frac{\partial p}{\partial x}\frac{h+2\alpha_h\mu}{h+\mu(\alpha_h+\alpha_s)}Z^2 + U\frac{h+\alpha_h\mu}{h+\mu(\alpha_h+\alpha_s)}Z$$

$$-\frac{h}{2\mu}\frac{\partial p}{\partial x}\frac{\alpha_s\mu(h+2\alpha_h\mu)}{h+\mu(\alpha_h+\alpha_s)}Z\Big|_0^h$$

$$= \left(\frac{h^3}{6\mu}\frac{\partial p}{\partial x} - \frac{h^3}{4\mu}\frac{\partial p}{\partial x}\frac{h+2\alpha_h\mu}{h+\mu(\alpha_h+\alpha_s)} - \frac{h^2}{2\mu}\frac{\partial p}{\partial x}\frac{\alpha_s\mu(h+2\alpha_h\mu)}{h+\mu(\alpha_h+\alpha_s)}\right)$$

$$+\left(U\frac{h^2+h\alpha_h\mu}{h+\mu(\alpha_h+\alpha_s)} - \frac{U}{2}\frac{h^2}{h+\mu(\alpha_h+\alpha_s)}\right)$$

$\int_0^h u\,dz = (a)+(b)$, dimana

$$(a) = \left(\frac{2h^3}{12\mu} - \frac{3h^3}{12\mu}\frac{h+2\alpha_h\mu}{h+\mu(\alpha_h+\alpha_s)} - \frac{6h^2}{12\mu}\frac{\alpha_s\mu(h+2\alpha_h\mu)}{h+\mu(\alpha_h+\alpha_s)}\right)\frac{\partial p}{\partial x}$$

$$(b) = \left(U\frac{h^2+h\alpha_h\mu}{h+\mu(\alpha_h+\alpha_s)} - \frac{U}{2}\frac{h^2}{h+\mu(\alpha_h+\alpha_s)}\right)$$

- Mencari (a)

$$= \frac{h^3}{12\mu}\frac{\partial p}{\partial x}\left\{2 - \frac{3\left(h + 2\alpha_h\mu\right)}{h + \mu\left(\alpha_h + \alpha_s\right)} - \frac{6\alpha_s\mu\left(h + 2\alpha_h\mu\right)}{h(h + \mu\left(\alpha_h + \alpha_s\right))}\right\}$$

$$= \frac{h^3}{12\mu}\frac{\partial p}{\partial x}\left\{\frac{2\left[h\left(h + \mu\left(\alpha_h + \alpha_s\right)\right)\right] - 3h\left(h + 2\alpha_h\mu\right) - 6\alpha_s\mu\left(h + 2\alpha_h\mu\right)}{h(h + \mu\left(\alpha_h + \alpha_s\right))}\right\}$$

$$= \frac{h^3}{12\mu}\frac{\partial p}{\partial x}\left\{\frac{2h^2 + 2h\mu\alpha_h + 2h\mu\alpha_s - 3h^2 - 6h\mu\alpha_h - 6h\alpha_s\mu - 12\mu^2\alpha_h\alpha_s}{h(h + \mu\left(\alpha_h + \alpha_s\right))}\right\}$$

$$= \frac{h^3}{12\mu}\frac{\partial p}{\partial x}\left\{\frac{-h^2 - 4h\mu\alpha_h - 4h\mu\alpha_s - 12\mu^2\alpha_h\alpha_s}{h(h + \mu\left(\alpha_h + \alpha_s\right))}\right\}$$

$$\therefore (a) = -\frac{h^3}{12\mu}\frac{\partial p}{\partial x}\left\{\frac{h^2 + 4h\mu\left(\alpha_h + \alpha_s\right) + 12\mu^2\alpha_h\alpha_s}{h(h + \mu\left(\alpha_h + \alpha_s\right))}\right\}$$

- Mencari (b)

$$= U\left(\frac{h^2 + h\alpha_h\mu}{h + \mu\left(\alpha_h + \alpha_s\right)} - \frac{h^2}{2(h + \mu\left(\alpha_h + \alpha_s\right))}\right)$$

$$= U\left(\frac{2h^2 + 2h\alpha_h\mu - h^2}{2\left(h + \mu\left(\alpha_h + \alpha_s\right)\right)}\right)$$

$$\therefore (b) = \frac{U}{2}\left(\frac{h^2 + 2h\alpha_h\mu}{h + \mu\left(\alpha_h + \alpha_s\right)}\right)$$

$$\therefore \int_0^h u\,dz = (a) + (b)$$

$$\int_0^h u\,dz = -\frac{\partial}{\partial x}\left(\frac{h^3}{12\mu}\frac{\partial p}{\partial x}\frac{h^2 + 4h\mu\left(\alpha_h + \alpha_s\right) + 12\mu^2\alpha_h\alpha_s}{h\left(h + \mu\left(\alpha_h + \alpha_s\right)\right)}\right) + \frac{U}{2}\frac{\partial}{\partial x}\left(\frac{h^2 + 2h\alpha_h\mu}{h + \mu\left(\alpha_h + \alpha_s\right)}\right) \quad (7)$$

Untuk $q_x = -u(h)\dfrac{\partial h}{\partial x}$

$$u(h) = \frac{1}{2\mu}\frac{\partial p}{\partial x}h^2 - \left(\frac{h}{2\mu}\frac{\partial p}{\partial x}\frac{h+2\alpha_h\mu}{h+\mu(\alpha_h+\alpha_s)} + \frac{U}{h+\mu(\alpha_h+\alpha_s)}\right)h + U\frac{h+\alpha_h\mu}{h+\mu(\alpha_h+\alpha_s)}$$
$$-\frac{h}{2\mu}\frac{\partial p}{\partial x}\frac{\alpha_s\mu(h+2\alpha_h\mu)}{h+\mu(\alpha_h+\alpha_s)}$$

$$-u(h)\frac{\partial h}{\partial x} = -\frac{1}{2\mu}\frac{\partial p}{\partial x}h^2 + \left(\frac{h}{2\mu}\frac{\partial p}{\partial x}\frac{h+2\alpha_h\mu}{h+\mu(\alpha_h+\alpha_s)} + \frac{U}{h+\mu(\alpha_h+\alpha_s)}\right)h - U\frac{h+\alpha_h\mu}{h+\mu(\alpha_h+\alpha_s)}$$
$$+\frac{h}{2\mu}\frac{\partial p}{\partial x}\frac{\alpha_s\mu(h+2\alpha_h\mu)}{h+\mu(\alpha_h+\alpha_s)}$$

$-u(h)\dfrac{\partial h}{\partial x} = (c)+(d)$, dimana

$$(c) = -\frac{h^2}{2\mu}\frac{\partial p}{\partial x} + \frac{h^2}{2\mu}\frac{\partial p}{\partial x}\frac{h+2\alpha_h\mu}{h+\mu(\alpha_h+\alpha_s)} + \frac{h}{2\mu}\frac{\partial p}{\partial x}\frac{\alpha_s\mu(h+2\alpha_h\mu)}{h+\mu(\alpha_h+\alpha_s)}$$

$$(d) = \frac{Uh}{h+\mu(\alpha_h+\alpha_s)} - U\frac{h+\alpha_h\mu}{h+\mu(\alpha_h+\alpha_s)}$$

- Mencari (c)

$$= \left(\frac{-h^2\left(h+\mu(\alpha_h+\alpha_s)\right) + h^3 + 2h^2\alpha_h\mu + h^2\alpha_s\mu + 2h\alpha_h\alpha_s\mu^2}{2\mu\left(h+\mu(\alpha_h+\alpha_s)\right)}\right)\frac{\partial p}{\partial x}$$

$$= \left(\frac{-h^3 - h^2\alpha_h\mu - h^2\alpha_s\mu + h^3 + 2h^2\alpha_h\mu + h^2\alpha_s\mu + 2h\alpha_h\alpha_s\mu^2}{2\mu\left(h+\mu(\alpha_h+\alpha_s)\right)}\right)\frac{\partial p}{\partial x}$$

$$= \left(\frac{h^2\alpha_h\mu + 2h\alpha_h\alpha_s\mu^2}{2\mu\left(h+\mu(\alpha_h+\alpha_s)\right)}\right)\frac{\partial p}{\partial x}$$

$$\therefore (c) = \frac{h}{2\mu}\frac{\partial p}{\partial x}\frac{h\alpha_h\mu + 2\alpha_h\alpha_s\mu^2}{h+\mu(\alpha_h+\alpha_s)}$$

- Mencari (d)

$$= \frac{Uh}{h+\mu\left(\alpha_h+\alpha_s\right)} - \frac{Uh}{h+\mu\left(\alpha_h+\alpha_s\right)} - \frac{U\alpha_h\mu}{h+\mu\left(\alpha_h+\alpha_s\right)}$$

$$\therefore \left(d\right) = -U\,\frac{\alpha_h\mu}{h+\mu\left(\alpha_h+\alpha_s\right)}$$

$$-u(h)\frac{\partial h}{\partial x} = (c)+(d)$$

$$\therefore u(h)\frac{\partial h}{\partial x} = \frac{h}{2\mu}\frac{\partial p}{\partial x}\frac{\partial h}{\partial x}\frac{h\alpha_h\mu+2\alpha_h\alpha_s\mu^2}{h+\mu\left(\alpha_h+\alpha_s\right)} - U\,\frac{\alpha_h\mu}{h+\mu\left(\alpha_h+\alpha_s\right)}\frac{\partial h}{\partial x} \qquad (8)$$

$$\therefore q_x = (7)+(8)$$

$$\therefore q_x = -\frac{\partial}{\partial x}\left(\frac{h^3}{12\mu}\frac{\partial p}{\partial x}\frac{h^2+4h\mu\left(\alpha_h+\alpha_s\right)+12\mu^2\alpha_h\alpha_s}{h\left(h+\mu\left(\alpha_h+\alpha_s\right)\right)}\right) + \frac{U}{2}\frac{\partial}{\partial x}\left(\frac{h^2+2h\alpha_h\mu}{2\left(h+\mu\left(\alpha_h+\alpha_s\right)\right)}\right)$$
$$+\frac{h}{2\mu}\frac{\partial p}{\partial x}\frac{\partial h}{\partial x}\frac{h\alpha_h\mu+2\alpha_h\alpha_s\mu^2}{h+\mu\left(\alpha_h+\alpha_s\right)} - U\,\frac{\alpha_h\mu}{h+\mu\left(\alpha_h+\alpha_s\right)}\frac{\partial h}{\partial x}$$

(9)

Laju aliran pada arah y adalah

$$v = \frac{1}{2\mu}\frac{\partial p}{\partial y}Z^2 - \left(\frac{h}{2\mu}\frac{\partial p}{\partial y}\frac{h+2\alpha_h\mu}{h+\mu\left(\alpha_h+\alpha_s\right)}\right)Z - \left(\frac{h}{2\mu}\frac{\partial p}{\partial y}\frac{\alpha_s\mu\left(h+2\alpha_h\mu\right)}{h+\mu\left(\alpha_h+\alpha_s\right)}\right)$$

$$q_y = \int_0^h v\,dz - v(h)\frac{\partial h}{\partial y} \tag{10}$$

Untuk $q_y = \int_0^h v\,dz$

$$= \int_0^h \frac{1}{2\mu}\frac{\partial p}{\partial y}Z^2 - \left(\frac{h}{2\mu}\frac{\partial p}{\partial y}\frac{h+2\alpha_h\mu}{h+\mu\left(\alpha_h+\alpha_s\right)}\right)Z - \left(\frac{h}{2\mu}\frac{\partial p}{\partial y}\frac{\alpha_s\mu\left(h+2\alpha_h\mu\right)}{h+\mu\left(\alpha_h+\alpha_s\right)}\right)dz$$

$$= \frac{1}{6\mu}\frac{\partial p}{\partial y}Z^3 - \frac{h}{4\mu}\frac{\partial p}{\partial y}\frac{h+2\alpha_h\mu}{h+\mu\left(\alpha_h+\alpha_s\right)}Z^2 - \frac{h}{2\mu}\frac{\partial p}{\partial y}\frac{\alpha_s\mu\left(h+2\alpha_h\mu\right)}{h+\mu\left(\alpha_h+\alpha_s\right)}\Bigg|_0^h$$

$$= \left(\frac{h^3}{6\mu}\frac{\partial p}{\partial y} - \frac{h^3}{4\mu}\frac{\partial p}{\partial y}\frac{h+2\alpha_h\mu}{h+\mu\left(\alpha_h+\alpha_s\right)} - \frac{h^2}{2\mu}\frac{\partial p}{\partial y}\frac{\alpha_s\mu\left(h+2\alpha_h\mu\right)}{h+\mu\left(\alpha_h+\alpha_s\right)}\right)$$

$$= \left(\frac{2h^3}{12\mu} - \frac{3h^3}{12\mu}\frac{h+2\alpha_h\mu}{h+\mu\left(\alpha_h+\alpha_s\right)} - \frac{6h^2}{12\mu}\frac{\alpha_s\mu\left(h+2\alpha_h\mu\right)}{h+\mu\left(\alpha_h+\alpha_s\right)}\right)\frac{\partial p}{\partial y}$$

$$= \frac{h^3}{12\mu}\frac{\partial p}{\partial y}\left\{2 - \frac{3\left(h+2\alpha_h\mu\right)}{h+\mu\left(\alpha_h+\alpha_s\right)} - \frac{6\alpha_s\mu\left(h+2\alpha_h\mu\right)}{h(h+\mu\left(\alpha_h+\alpha_s\right))}\right\}$$

$$= \frac{h^3}{12\mu}\frac{\partial p}{\partial y}\left\{\frac{2\left[h\left(h+\mu\left(\alpha_h+\alpha_s\right)\right)\right] - 3h\left(h+2\alpha_h\mu\right) - 6\alpha_s\mu\left(h+2\alpha_h\mu\right)}{h(h+\mu\left(\alpha_h+\alpha_s\right))}\right\}$$

$$= \frac{h^3}{12\mu}\frac{\partial p}{\partial y}\left\{\frac{2h^2 + 2h\mu\alpha_h + 2h\mu\alpha_s - 3h^2 - 6h\mu\alpha_h - 6h\alpha_s\mu - 12\mu^2\alpha_h\alpha_s}{h(h+\mu\left(\alpha_h+\alpha_s\right))}\right\}$$

$$= \frac{h^3}{12\mu}\frac{\partial p}{\partial y}\left\{\frac{-h^2 - 4h\mu\alpha_h - 4h\mu\alpha_s - 12\mu^2\alpha_h\alpha_s}{h(h+\mu\left(\alpha_h+\alpha_s\right))}\right\}$$

$$\therefore q_y = -\frac{h^3}{12\mu}\frac{\partial p}{\partial y}\left\{\frac{h^2 + 4h\mu\left(\alpha_h+\alpha_s\right) + 12\mu^2\alpha_h\alpha_s}{h\left(h+\mu\left(\alpha_h+\alpha_s\right)\right)}\right\} \tag{11}$$

Untuk $q_y = -v(h)\dfrac{\partial h}{\partial y}$

$$v(h) = \frac{1}{2\mu}\frac{\partial p}{\partial y}h^2 - \left(\frac{h}{2\mu}\frac{\partial p}{\partial y}\frac{h+2\alpha_h\mu}{h+\mu\left(\alpha_h+\alpha_s\right)}\right)h - \frac{h}{2\mu}\frac{\partial p}{\partial y}\frac{\alpha_s\mu\left(h+2\alpha_h\mu\right)}{h+\mu\left(\alpha_h+\alpha_s\right)}$$

$$-v(h)\frac{\partial h}{\partial y} = -\frac{1}{2\mu}\frac{\partial p}{\partial y}h^2 + \left(\frac{h}{2\mu}\frac{\partial p}{\partial y}\frac{h+2\alpha_h\mu}{h+\mu\left(\alpha_h+\alpha_s\right)}\right)h + \frac{h}{2\mu}\frac{\partial p}{\partial y}\frac{\alpha_s\mu\left(h+2\alpha_h\mu\right)}{h+\mu\left(\alpha_h+\alpha_s\right)}$$

$$= \left(\frac{-h^2\left(h+\mu\left(\alpha_h+\alpha_s\right)\right)+h^3+2h^2\alpha_h\mu+h^2\alpha_s\mu+2h\alpha_h\alpha_s\mu^2}{2\mu\left(h+\mu\left(\alpha_h+\alpha_s\right)\right)}\right)\frac{\partial p}{\partial y}$$

$$= \left(\frac{-h^3-h^2\alpha_h\mu-h^2\alpha_s\mu+h^3+2h^2\alpha_h\mu+h^2\alpha_s\mu+2h\alpha_h\alpha_s\mu^2}{2\mu\left(h+\mu\left(\alpha_h+\alpha_s\right)\right)}\right)\frac{\partial p}{\partial y}$$

$$= \left(\frac{h^2\alpha_h\mu+2h\alpha_h\alpha_s\mu^2}{2\mu\left(h+\mu\left(\alpha_h+\alpha_s\right)\right)}\right)\frac{\partial p}{\partial y}$$

$$\therefore v(h)\frac{\partial h}{\partial y} = \frac{h}{2\mu}\frac{\partial p}{\partial y}\frac{\partial h}{\partial y}\frac{h\alpha_h\mu+2\alpha_h\alpha_s\mu^2}{h+\mu\left(\alpha_h+\alpha_s\right)} \qquad (12)$$

$$\therefore q_y = (9)+(10)$$

$$\therefore q_y = -\frac{\partial}{\partial y}\left(\frac{h^3}{12\mu}\frac{\partial p}{\partial y}\frac{h^2+4h\mu\left(\alpha_h+\alpha_s\right)+12\mu^2\alpha_h\alpha_s}{h\left(h+\mu\left(\alpha_h+\alpha_s\right)\right)}\right) + \frac{h}{2\mu}\frac{\partial p}{\partial y}\frac{\partial h}{\partial y}\frac{h\alpha_h\mu+2\alpha_h\alpha_s\mu^2}{h+\mu\left(\alpha_h+\alpha_s\right)}$$

Persamaan Navier *Slip* dua dimensi dengan dua *slip* yaitu

$$\frac{\partial}{\partial x}\left(\frac{h^3}{12\mu}\frac{\partial p}{\partial x}\frac{h^2+4h\mu\left(\alpha_h+\alpha_s\right)+12\mu^2\alpha_h\alpha_s}{h\left(h+\mu\left(\alpha_h+\alpha_s\right)\right)}\right) + \frac{\partial}{\partial y}\left(\frac{h^3}{12\mu}\frac{\partial p}{\partial y}\frac{h^2+4h\mu\left(\alpha_h+\alpha_s\right)+12\mu^2\alpha_h\alpha_s}{h\left(h+\mu\left(\alpha_h+\alpha_s\right)\right)}\right) =$$

$$+\frac{U}{2}\frac{\partial}{\partial x}\left(\frac{h^2+2h\alpha_h\mu}{h+\mu\left(\alpha_h+\alpha_s\right)}\right) - U\frac{\alpha_h\mu}{h+\mu\left(\alpha_h+\alpha_s\right)}\frac{\partial h}{\partial x} + \frac{h}{2\mu}\frac{h\alpha_h\mu+2\alpha_h\alpha_s\mu^2}{h+\mu\left(\alpha_h+\alpha_s\right)}\left(\frac{\partial p}{\partial x}\frac{\partial h}{\partial x}+\frac{\partial p}{\partial y}\frac{\partial h}{\partial y}\right) + \frac{\partial h}{\partial t}$$

A. 2. Diskretisasi Navier *Slip* 2D – 2 *Slip* tanpa Efek *Squeeze*

Persamaan Navier *Slip* dua dimensi dengan dua *slip* yaitu

$$
\begin{aligned}
&\frac{\partial}{\partial x}\left(\frac{h^3}{12\mu}\frac{\partial p}{\partial x}\frac{h^2+4h\mu\left(\alpha_h+\alpha_s\right)+12\mu^2\alpha_h\alpha_s}{h\left(h+\mu\left(\alpha_h+\alpha_s\right)\right)}\right)+\\
&\frac{\partial}{\partial y}\left(\frac{h^3}{12\mu}\frac{\partial p}{\partial y}\frac{h^2+4h\mu\left(\alpha_h+\alpha_s\right)+12\mu^2\alpha_h\alpha_s}{h\left(h+\mu\left(\alpha_h+\alpha_s\right)\right)}\right)=\\
&+\frac{U}{2}\frac{\partial}{\partial x}\left(\frac{h^2+2h\alpha_h\mu}{h+\mu\left(\alpha_h+\alpha_s\right)}\right)-U\frac{\alpha_h\mu}{h+\mu\left(\alpha_h+\alpha_s\right)}\frac{\partial h}{\partial x}+\\
&\frac{h}{2\mu}\frac{h\alpha_h\mu+2\alpha_h\alpha_s\mu^2}{h+\mu\left(\alpha_h+\alpha_s\right)}\left(\frac{\partial p}{\partial x}\frac{\partial h}{\partial x}+\frac{\partial p}{\partial y}\frac{\partial h}{\partial y}\right)+\frac{\partial h}{\partial t}
\end{aligned}
\qquad (1)
$$

Navier *slip* tanpa efek *squeeze* yaitu

$$
\begin{aligned}
&\frac{\partial}{\partial x}\left(\frac{h^3}{12\mu}\frac{\partial p}{\partial x}\frac{h^2+4h\mu\left(\alpha_h+\alpha_s\right)+12\mu^2\alpha_h\alpha_s}{h\left(h+\mu\left(\alpha_h+\alpha_s\right)\right)}\right)+\\
&\frac{\partial}{\partial y}\left(\frac{h^3}{12\mu}\frac{\partial p}{\partial y}\frac{h^2+4h\mu\left(\alpha_h+\alpha_s\right)+12\mu^2\alpha_h\alpha_s}{h\left(h+\mu\left(\alpha_h+\alpha_s\right)\right)}\right)=\frac{U}{2}\frac{\partial}{\partial x}\left(\frac{h^2+2h\alpha_h\mu}{h+\mu\left(\alpha_h+\alpha_s\right)}\right)
\end{aligned}
\qquad (2)
$$

Dengan asumsi *slip* terjadi pada kedua bagian yaitu *shaft* dan *housing*.

$$
\begin{aligned}
&\frac{\partial}{\partial x}\left(h^3\frac{\partial p}{\partial x}\frac{h^2+4h\mu\left(\alpha_h+\alpha_s\right)+12\mu^2\alpha_h\alpha_s}{h\left(h+\mu\left(\alpha_h+\alpha_s\right)\right)}\right)+\\
&\frac{\partial}{\partial y}\left(h^3\frac{\partial p}{\partial y}\frac{h^2+4h\mu\left(\alpha_h+\alpha_s\right)+12\mu^2\alpha_h\alpha_s}{h\left(h+\mu\left(\alpha_h+\alpha_s\right)\right)}\right)=6\mu U\frac{\partial}{\partial x}\left(\frac{h^2+2h\alpha_h\mu}{h+\mu\left(\alpha_h+\alpha_s\right)}\right)
\end{aligned}
\qquad (3)
$$

Persamaan umum sesuai dengan persamaan yang didapat sebelumnya, persamaan (3). Persamaan umum diintegralkan seluruh *control volume.*

$$\int_s^n\int_e^w \frac{\partial}{\partial x}\left(h^3\frac{\partial p}{\partial x}\left(\frac{h^2+4h\mu\left(\alpha_h+\alpha_s\right)+12\mu^2\alpha_h\alpha_s}{h\left(h+\mu\left(\alpha_h+\alpha_s\right)\right)}\right)\right)dxdy+$$

$$\int_s^n\int_e^w \frac{\partial}{\partial y}\left(h^3\frac{\partial p}{\partial y}\left(\frac{h^2+4h\mu\left(\alpha_h+\alpha_s\right)+12\mu^2\alpha_h\alpha_s}{h\left(h+\mu\left(\alpha_h+\alpha_s\right)\right)}\right)\right)dxdy=$$

$$\int_s^n\int_e^w \frac{\partial}{\partial x}\left(6\mu U\left(\frac{h^2+2h\alpha_h\mu}{h+\mu\left(\alpha_h+\alpha_s\right)}\right)\right)dxdy$$

$$\int_s^n\int_e^w \frac{\partial}{\partial x}\left(K\frac{\partial p}{\partial x}\right)dxdy+\int_s^n\int_e^w \frac{\partial}{\partial y}\left(K\frac{\partial p}{\partial y}\right)dxdy=6\mu U\int_s^n\int_e^w \frac{\partial}{\partial x}\left(C\right)dxdy \qquad (4)$$

Dimana K dan C adalah variabel untuk menyederhanakan persamaan (4) dan didefinisikan sebagai berikut:

$$K=h^3\left(\frac{h^2+4h\mu\left(\alpha_h+\alpha_s\right)+12\mu^2\alpha_h\alpha_s}{h\left(h+\mu\left(\alpha_h+\alpha_s\right)\right)}\right) \qquad (5)$$

$$C=h\left(\frac{h+2\alpha_h\mu}{h+\mu\left(\alpha_h+\alpha_s\right)}\right) \qquad (6)$$

Sehingga integral persamaan umum menjadi:

$$\left(K\frac{\partial p}{\partial x}\right)_e\Delta y-\left(K\frac{\partial p}{\partial x}\right)_w\Delta y+\left(K\frac{\partial p}{\partial y}\right)_n\Delta x-\left(K\frac{\partial p}{\partial y}\right)_s\Delta x=$$

$$6\mu U\left[\left(C\right)_e\Delta y-\left(C\right)_w\Delta y\right] \qquad (7)$$

$$K_e\frac{p_E-p_P}{\Delta x}\Delta y-K_w\frac{p_P-p_W}{\Delta x}\Delta y+K_n\frac{p_N-p_P}{\Delta y}\Delta x-K_s\frac{p_P-p_S}{\Delta y}\Delta x=$$

$$3\mu U\left[\left(C\right)_E\Delta y-\left(C\right)_W\Delta y\right] \qquad (8)$$

Diskretisasi akhir

$$a_P P_P = a_E P_E + a_W P_W + a_N P_N + a_S P_S + S_c \qquad (9)$$

Dengan koefisien

$$a_E = \frac{k_e}{\Delta x}\Delta y \qquad k_e = \frac{2K_E K_P}{K_E + K_P}$$

$$a_W = \frac{k_w}{\Delta x}\Delta y \qquad k_w = \frac{2K_W K_P}{K_W + K_P}$$

$$a_N = \frac{k_n}{\Delta y}\Delta x \qquad k_n = \frac{2K_N K_P}{K_N + K_P}$$

$$a_S = \frac{k_s}{\Delta y}\Delta x \qquad k_s = \frac{2K_S K_P}{K_S + K_P}$$

$$a_P = a_E + a_W + a_N + a_S \qquad (10)$$

$$S_c = 3\mu U\left[(C)_W - (C)_E\right]\Delta y \qquad (11)$$

Jika *slip* hanya terjadi pada bagian *housing*, maka persamaan (2) akan tereduksi menjadi

$$\frac{\partial}{\partial x}\left(\frac{h^3}{12\mu}\frac{\partial p}{\partial x}\frac{h+4\mu\alpha_h}{h+\mu\alpha_h}\right)+\frac{\partial}{\partial y}\left(\frac{h^3}{12\mu}\frac{\partial p}{\partial y}\frac{h+4\mu\alpha_h}{h+\mu\alpha_h}\right)=\frac{U}{2}\frac{\partial}{\partial x}\left(\frac{h^2+2h\alpha_h\mu}{h+\mu\alpha_h}\right) \tag{12}$$

Kemudian

$$\frac{\partial}{\partial x}\left(h^3\frac{\partial p}{\partial x}\frac{h+4\mu\alpha_h}{h+\mu\alpha_h}\right)+\frac{\partial}{\partial y}\left(h^3\frac{\partial p}{\partial y}\frac{h+4\mu\alpha_h}{h+\mu\alpha_h}\right)=6\mu U\frac{\partial}{\partial x}\left(\frac{h^2+2h\alpha_h\mu}{h+\mu\alpha_h}\right) \tag{13}$$

Persamaan umum sesuai dengan persamaan yang didapat sebelumnya, persamaan (13). Persamaan umum diintegralkan seluruh *control volume.*

$$\begin{aligned}&\int_s^n\int_e^w\frac{\partial}{\partial x}\left(h^3\frac{\partial p}{\partial x}\left(\frac{h+4\mu\alpha_h}{h+\mu\alpha_h}\right)\right)dxdy+\int_s^n\int_e^w\frac{\partial}{\partial y}\left(h^3\frac{\partial p}{\partial y}\left(\frac{h+4\mu\alpha_h}{h+\mu\alpha_h}\right)\right)dxdy=\\&\int_s^n\int_e^w\frac{\partial}{\partial x}\left(6\mu U\left(\frac{h^2+2h\alpha_h\mu}{h+\mu\alpha_h}\right)\right)dxdy\end{aligned} \tag{14}$$

$$\int_s^n\int_e^w\frac{\partial}{\partial x}\left(K\frac{\partial p}{\partial x}\right)dxdy+\int_s^n\int_e^w\frac{\partial}{\partial y}\left(K\frac{\partial p}{\partial y}\right)dxdy=6\mu U\int_s^n\int_e^w\frac{\partial}{\partial x}(C)\,dxdy \tag{15}$$

Dimana K dan C adalah variabel untuk menyederhanakan persamaan (15) dan didefinisikan sebagai berikut:

$$K=h^3\left(\frac{h+4\mu\alpha_h}{h+\mu\alpha_h}\right) \tag{16}$$

$$C=h\left(\frac{h+2\alpha_h\mu}{h+\mu\alpha_h}\right) \tag{17}$$

Sehingga integral persamaan umum menjadi:

$$\begin{aligned}&\left(K\frac{\partial p}{\partial x}\right)_e\Delta y-\left(K\frac{\partial p}{\partial x}\right)_w\Delta y+\left(K\frac{\partial p}{\partial y}\right)_n\Delta x-\left(K\frac{\partial p}{\partial y}\right)_s\Delta x=\\&6\mu U\left[(C)_e\Delta y-(C)_w\Delta y\right]\end{aligned} \tag{18}$$

$$K_e \frac{p_E - p_P}{\Delta x}\Delta y - K_w \frac{p_P - p_W}{\Delta x}\Delta y + K_n \frac{p_N - p_P}{\Delta y}\Delta x - K_s \frac{p_P - p_S}{\Delta y}\Delta x =$$

$$3\mu U\left[(C)_E \Delta y - (C)_W \Delta y\right] \tag{19}$$

Diskretisasi akhir

$$a_P P_P = a_E P_E + a_W P_W + a_N P_N + a_S P_S + S_c \tag{20}$$

Dengan koefisien

$$a_E = \frac{k_e}{\Delta x}\Delta y \qquad k_e = \frac{2K_E K_P}{K_E + K_P}$$

$$a_W = \frac{k_w}{\Delta x}\Delta y \qquad k_w = \frac{2K_W K_P}{K_W + K_P}$$

$$a_N = \frac{k_n}{\Delta y}\Delta x \qquad k_n = \frac{2K_N K_P}{K_N + K_P}$$

$$a_S = \frac{k_s}{\Delta y}\Delta x \qquad k_s = \frac{2K_S K_P}{K_S + K_P}$$

$$a_P = a_E + a_W + a_N + a_S \tag{21}$$

$$S_c = 3\mu U\left[(C)_W - (C)_E\right]\Delta y \tag{22}$$