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HETEROSCEDASTIC TIME SERIES MODEL BY WAVELET TRANSFORM

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Abstract. Box-Jenkins is the best method for stationary time series modeling. When variance varies over time, it is proposed to use ARCH model to capture time series structure (Engle, 1982). In 1986, Bollerslev generalized it into GARCH model. The standard approach of models is introducing an exogenous variable along with some assumptions. This paper is proposed an alternative solution when exogenous variable unfulfilled these assumptions. Discrete wavelet transform can be used to analyze time series structure when the sample size is integer power of 2. When sample size is arbitrarily, it's proposed to use undecimated wavelet transform.

Keywords and Phrases : Heteroscedastic, ARCH model, GARCH Model, Wavelet Transform

1. INTRODUCTION

Volatility in time series data is indicated by changing of variance value over time. It means there is a heteroscedasticity property in data. According to this condition, the data can not be modeled by Box-Jenkins method directly. The early heteroscedastic model was proposed by Engel [5] in 1982, which is called as Autoregressive Conditional Heteroscedasticity (ARCH) model. Heteroscedastic properties in data are captured by AR(p) model of error component.

$$\varepsilon_t = \sigma_t v_t,$$
$$\sigma_t^2 = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \ldots + \alpha_1 \varepsilon_{t-p}^2 \qquad (1)$$

where $\{v_t\}$is a sequence of iid with mean 0 and variance 1, $\alpha_0>0$, and $\alpha_i \geq 0$ for $i>0$. In practice, v_t is usually assumed to follow the standard normal or a standardized student-t. The model (1) was developed by Bollerslev [2] in 1986 along with assumption that σ_t^2 follows an ARMA(p,q) model. This paper does not cover complete solution of ARCH/GARCH model, but will gives an alternative solution when there is a violation of v_t assumption. However the study of wavelet method is included in nonparametric modeling which free of distribution assumption.

2. WAVELET AND FILTERING

Wavelet is a small wave function that can build an orthonormal basis for $L_2(R)$, so that every function $f \in L_2(R)$ can be expressed as linear combination of wavelets [4]

$$\begin{aligned} f(t) &= \sum_{k \in Z} c_{J,k}\phi_{J,k}(t) + \sum_{j \le J}\sum_{k \in Z} w_{j,k}\psi_{j,k}(t) \\ &= S_J + D_J + D_{J-1} + \dots + D_1 \end{aligned} \tag{2}$$

where ϕ and ψ is a father and mother wavelet respectively with dilation and translation indexes

$$\psi_{j,k}(t) = 2^{-j/2}\psi(2^{-j}t - k) \tag{3a}$$

$$\phi_{j,k}(t) = 2^{-j/2}\phi(2^{-j}t - k). \tag{3b}$$

In discrete version wavelet can construct an orthonormal filter matrix so that every discrete realization of $f \in L_2(R)$ can be decomposed into scaling component or smooth component (S) and detail components (D) [8].

Let $\mathbf{h}=[h_0, h_1, \dots, h_{L-1}]$ a wavelet filter then scaling filter $\mathbf{g}$ can be derived from $\mathbf{h}$ by formulation (4)

$$g_i = (-1)^{i+1} h_{L-1-i}\,,\ i=0,1,\dots, L-1 \tag{4}$$

For example the Haar filter $\mathbf{h}$ and its scaling filter will be formed as equation (5)

$$\mathbf{h}=[\tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}}] \text{ and } \mathbf{g}=[\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}] \tag{5}$$

Let $\{Z_t\}$ is time series from discrete time realization of $f \in L_2(R)$ with $t=1,2,\dots, N$, $N=2^J$, then the coefficient $w_{j,k}$ and $c_{J,k}$ in equation (2) can be computed by discrete wavelet transform (DWT) as shown in (6). Here, $\mathbf{H}$ is a filter matrix of NxN

$$\mathbf{W}=\mathbf{H}.\mathbf{Z}=\begin{bmatrix} h_1^{(1)} & h_0^{(1)} & 0 & \dots & 0 & h_{L-1}^{(1)} & \dots & h_2^{(1)} \\ h_3^1 & h_2^1 & h_1^1 & h_0^1 & 0 & \dots & h_5^1 & h_4^1 \\ \vdots & & & & & & & \\ 0 & 0 & \dots & h_{L-1}^1 & h_{L-2}^1 & \dots & h_1^1 & h_0^1 \\ h_3^{(2)} & h_2^{(2)} & h_1^{(2)} & h_0^{(2)} & \dots & h_{3L-2}^{(2)} & \dots & h_4^{(2)} \\ \vdots & & & & & & & \\ h_{2^J-1}^{(J)} & h_{2^J-2}^{(J)} & \dots & & & & h_1^{(J)} & h_0^{(J)} \\ g_{2^J-1}^{(J)} & g_{2^J-1}^{(J)} & \dots & & & & g_1^{(J)} & g_0^{(J)} \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \\ \vdots \\ \\ \\ \\ \\ Z_N \end{bmatrix} \tag{6}$$

Up-sampled version of $\mathbf{h}$ notion by $\mathbf{h_{up}}$ is constructed by inserting zero between non-zero value filter. The filter of high level ($j=2,3,\dots,J$) is gotten by convolution of $\mathbf{h_{up}}$ and $\mathbf{g}$ as shown in (7).

$$\mathbf{h}^{(j)}=(\mathbf{h}^{(j-1)})_{up}*\mathbf{g} \qquad (7)$$

For example in Haar case $\mathbf{h}^{(2)}=[\frac{1}{\sqrt{2}},0,-\frac{1}{\sqrt{2}}]*[\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}]=[\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2}]$

When the sample size is not the form of 2^J, $J\in Z$, the coefficient $\mathbf{w}_{j,k}$ and $\mathbf{c}_{J,k}$ can be computed by Undecimated Wavelet Transform (UDWT). The scenario of UDWT for j=1 can be shown in Figure 1. The wavelet coefficient $\mathbf{w}_{1,k}$ is resulted by convolution of time series **Z** and **h**. The first detail component $\mathbf{D}_1$ is resulted by convolution of $\mathbf{w}_{1,k}$ and **h'** where **h'** is time reverse version of **h**. The scaling coefficient $\mathbf{c}_{1,k}$ is resulted by convolution of **Z** and **g**. The first scaling coefficient is resulted by convolution of $\mathbf{c}_{1,k}$ and **g'** where **g'** is time reverse version of **g**. Furthermore $\hat{Z}=\mathbf{S}_1+\mathbf{D}_1$ will equal to Z regard to wavelet filtering.

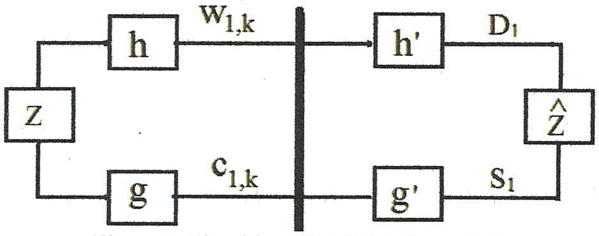

Figure 1. Algorithm of UDWT at level j=1

Higher level of UDWT can be constructed by split the scaling coefficient $\mathbf{c}_{j,k}$ into $\mathbf{c}_{j+1,k}$ and $\mathbf{w}_{j+1,k}$. The UDWT for j=2 can be shown in Figure 2. Furthermore $\hat{Z}=S_2+D_2+D_1$ will equal to Z regard to wavelet filtering.

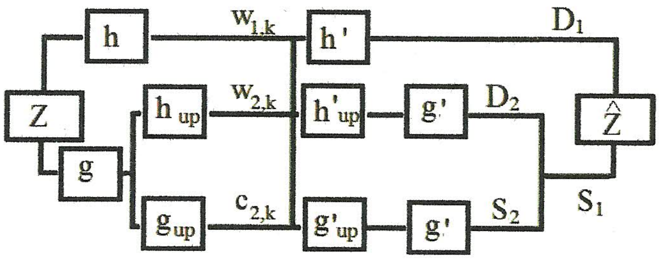

Figure 2. Algorithm of UDWT at level j=2

The number of DWT coefficients at level j+1 is a half of level j. In other hand, the number of UDWT coefficient always the same for all decomposition level. This property makes UDWT more powerful to analyze the time series than DWT. Furthermore, this paper will discuss wavelet base prediction of time series by UDWT only.

3. WAVELET BASE PREDICTION MODEL

Prediction of Z at time t+1 will be done refer to realization of Z in the past and wavelet coefficient which resulted from decomposition. Starck [9] propose the necessary wavelet coefficients at each level j which will be used for forecasting at time t+1 have the form $w_{j,N-2^j(k-1)}$ and $c_{J,N-2^j(k-1)}$. The forecasting formulation is expressed in equation (8)

$$\hat{Z}_{N+1} = \sum_{j=1}^{J}\sum_{k=1}^{A_j} \hat{a}_{j,k} w_{j,N-2^j(k-1)} + \sum_{k=1}^{A_{J+1}} \hat{a}_{J+1,k} c_{J,N-2^J(k-1)} \qquad (8)$$

The highest level of decomposition is indicated by J, and A_j is indicate the number of coefficients which chosen at level j. For example, if J=4 and A_j=2 for j=1,2,3,4 then (8) can be expressed as (9)

$$\begin{aligned}\hat{Z}_{N+1} &= \hat{a}_{1,1}w_{1,N} + \hat{a}_{1,2}w_{1,N-2} + \hat{a}_{2,1}w_{2,N} + \hat{a}_{2,2}w_{2,N-4} \\ &+ \hat{a}_{3,1}w_{3,N} + \hat{a}_{3,2}w_{3,N-8} + \hat{a}_{4,1}w_{4,N} + \hat{a}_{4,2}w_{4,N-16} \\ &+ \hat{a}_{5,1}c_{4,N} + \hat{a}_{5,2}c_{4,N-16}\end{aligned} \qquad (9)$$

Furthermore, least square method can be used for estimating coefficient $a_{i,k}$ in equation (8) and (9).

4. IMPLEMENTATION AND RESULT

The data of currency exchange from USD to IDR will be used for implementing the proposed method. The daily equivalent value of $1 to IDR along of 2003 year will be modeled according to equation (9). The statistic test will be appeared to check that the data is reasonable for this aim.

The actual data which is appeared in Figure 4 shows that the data comes from a non-stationary process. The Box-Jenkins standard method proposed to difference the data. The result of one lag data differencing can be shown in Figure 3, which gives a sign of heteroscedasticity feature. The ACF and PACF plot give a sign that there is neither AR nor MA which is significant. It looks like that the data can be modeled as $Z_t=\varepsilon_t$, where ε_t are not normally distributed. The Ljung-Box test for $\{\varepsilon_t\}$ and $\{\varepsilon_t^2\}$ indicate that $\{\varepsilon_t\}$ are independent, but $\{\varepsilon_t^2\}$ are dependent. So, it can be concluded that the heterogeneity of variances are occurred. The GARCH(1,1) looks like as the nearest model for $\{Z_t\}$, but the Jarque-Bera test is not supporting the residual normality assumption. Finally, it is concluded that the standard ARIMA and GARCH models have been failed to capture the data pattern.

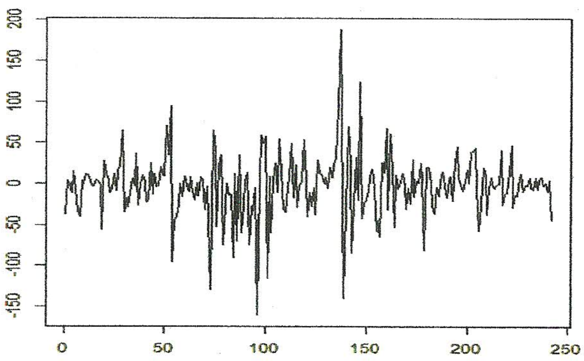

Figure 3. One Lag Differencing Value

As has been discussed above, there is a long way to reach final solution in parametric modeling. Next, it will be appeared the simpler way to make a prediction model in non-parametric sense, especially in wavelet based model. Although the wavelet computation theory is a complicated problem, but there are some software which make it easier. The step by step algorithm of modeling can be explained as follows

1. Exploring wavelet coefficients
   The Wavelet R-packaged which arranged by Aldrich [1] is used for exploring the UDWT wavelet coefficients with sample size N= 243 and decomposition level J= 4.

2. Collecting the selected coefficients
   For i in range 1 to N-16, there are defined the vectors of selected coefficients.
   w1=vector of wavelet coefficients at scale 1 with index of the form i+16
   w2= vector of wavelet coefficients at scale 1 with index of the form i+14
   w3= vector of wavelet coefficients at scale 2 with index of the form i+16
   w4= vector of wavelet coefficients at scale 2 with index of the form i+12
   w5=vector of wavelet coefficients at scale 3 with index of the form i+16
   w6= vector of wavelet coefficients at scale 3 with index of the form i+8
   w7= vector of wavelet coefficients at scale 4 with index of the form i+16
   w8= vector of wavelet coefficients at scale 4 with index of the form i
   c9= vector of scaling coefficients at scale 4 with index of the form i+16
   c10= vector of scaling coefficients at scale 4 with index of the form i

3. Calculating the parameter estimation of model
   The coefficients of equation (8) are computed regard to minimizing the sum squared of error. It can be solved by linear model in R packaged z~ w1+...+w8+c9+c10, where z is time series data with index ≥18. Furthermore, fitted values and residuals can be computed so that the MSE can be computed too.

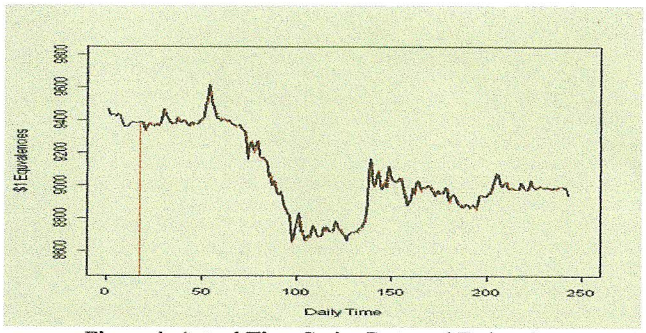

Figure 4. Actual Time Series Data and Estimation

The summary of parameter estimation gives the form of prediction model as (10)

$$\begin{aligned}\hat{z}_{n+1} = {} & 1.18w_{1,n} - 0.1469w_{1,n-2} + 0.7868w_{2,n} - 0.1006w_{2,n-4} \\ & + 0.9919w_{3,n} - 0.0240w_{3,n-8} + 1.152w_{4,n} + 0.0632w_{4,n-16} \\ & + 0.9841c_{4,n} + 0.0157c_{4,n-16}\end{aligned} \quad (10)$$

The time series plot of data and fitted values can be shown in Figure 4. The black line is a view of actual data and the red dash line is a view of fitted values. The mean square of error in this level is 4.14.

5. CONCLUDING REMARK

Wavelet transform, especially UDWT can be used for producing estimation model of time series. This modeling is simpler and easier to be implemented. The graphical views show that this method gives a good approximation. However a wide comparison to another methods and further analytical study must be done to make a comprehensives conclusion.

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R Code Listing

```
function (x,wv='haar',j=4)
{
n=length(x)
x.modwt=modwt(x,wv,j)
d1=x.modwt@W$W1
d2=x.modwt@W$W2
d3=x.modwt@W$W3
d4=x.modwt@W$W4
v4=x.modwt@V$V4
w1<-w2<-w3<-w4<-w5<-w6<-w7<-w8<-c9<-c10<-NULL
for (i in 1:(n-17)){
        w1<-c(w1,d1[i+16])
        w2<-c(w2,d1[i+14])
        w3<-c(w3,d2[i+16])
        w4<-c(w4,d2[i+12])
        w5<-c(w5,d3[i+16])
        w6<-c(w6,d3[i+8])
        w7<-c(w7,d4[i+16])
        w8<-c(w8,d4[i])
        c9<-c(c9,v4[i+16])
        c10<-c(c10,v4[i])
}
z=x[18:n]
lm.z=lm(z~-1+w1+w2+w3+w4+w5+w6+w7+w8+c9+c10)
koef<-lm.y$coeff
pred<-c(rep(0,17), lm.y$fitted)
ts.plot(z,xlim=c(0,250), ylim=c(8500,9800), xlab="", ylab="",
type= 'l')
par(new=T)
ts.plot(pred, xlim=c(0,250), xlab="Daily Time",ylab="$1
Equvalencies", ylim=c(8500,9800), col=2, lty=4)
return(lm.z)
}
```

DATA

```
> kurs2003
 9468 9431 9435 9435 9424 9440 9433 9400 9360 9364 9364 9376 9387 9390 9388
 9385 9390 9393 9392 9336 9364 9376 9380 9369 9363 9375 9367 9384 9405 9470
 9435 9417 9389 9378 9384 9381 9418 9392 9392 9402 9405 9383 9363 9388 9375
 9387 9383 9380 9400 9410 9419 9490 9525 9620 9525 9480 9440 9415 9415 9399
 9408 9406 9397 9405 9396 9376 9377 9362 9370 9374 9342 9339 9295 9165 9230
 9270 9218 9240 9275 9200 9175 9175 9161 9148 9058 9070 9000 9035 8975 8949
 8951 8965 8890 8863 8830 8825 8665 8670 8730 8779 8837 8721 8730 8670 8675
8700 8690 8745 8760 8730 8695 8690 8698 8747 8730 8753 8725 8723 8726 8780
8785 8745 8735 8708 8703 8666 8695 8709 8718 8718 8725 8720 8740 8747 8770
8801 8895 9083 9165 9025 8995 9065 9090 9005 8980 9013 8993 9118 9076 9053
9033 9025 9034 9053 9047 8989 8944 8880 8906 8920 8988 8957 9018 9035 8983
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8994 8988 8990 9003 9000 8970 8965 8941 8971 8955 8959 8960 8985 8991 8910
8930 8950 8950 8925 8889 8885 8870 8875 8890 8888 8871 8877 8886 8865 8893
8939 8945 8945 8937 8940 8958 8959 8998 9038 9083 9077 9020 8995 9015 9025
8988 8983 8990 8986 8981 8980 8980 9022 8997 8985 8974 8990 9037 9009 8996
8981 8988 9000 8991 8990 8988 8995 8990 8983 8990 8983 8988 8996 8994 8995
8986 8991 8947