BAB III

TRANSFORMASI HANKEL

3.1. Definisi Transformasi Hankel

Transformasi Hankel dari fungsi $f(x)$ didefinisikan dengan :

$$\bar{f}(\varphi) = H_v\left[f(x);\varphi\right] = \int_0^\infty x\ f(x)\ J_v(\varphi x)\ dx \quad , v \geq -\tfrac{1}{2}$$

dengan $$J_v(\varphi x) = \sum_{r=0}^{\infty} \frac{(-1)\ (\varphi x/2)^{2r+v}}{r!\ \Gamma(r+v+1)}$$

Transformasi Hankel dari fungsi $f(x)$ ada jika :

- $f(x)$ kontinu sepotong sepotong untuk setiap interval berhingga.
- $\int_{-\infty}^{\infty} |\ f(x)\ |\ dx$ konvergen sehingga seluruh fungsi $f(x)$ absolut integrabel.

sehingga terlihat ada kesamaan syarat antara Transformasi Hankel dan Transformasi Fourier.

Dalam penerapan Transformasi Fourier sering dijumpai $F_{(n)}(\varphi)$ dari fungsi $f(r)$ yang merupakan fungsi dari :

$$r = |\ x\ | = (x_1^2 + x_2^2 + \ldots + x_n^2)^{1/2}$$

akan diperlihatkan bahwa $F_{(n)}(\varphi)$ dapat dinyatakan dengan Transformasi Hankel dari fungsi $f(r)$.

Pertama dapat diambil dua dimensi dan jika diambil koordinat bidang polar (r,θ) dalam bidang X_1, X_2 dan (ρ,ϕ) dalam bidang φ_1,φ_2 misal :

$$x_1 = r \cos\theta \quad ; \quad \varphi_1 = \rho \cos\phi$$

$$x_2 = r \sin\theta \quad ; \quad \varphi_2 = \rho \sin\phi$$

$$x.\varphi = x_1 \varphi_1 + x_2 \varphi_2$$

$$= r\,\rho \cos\theta \cos\phi + r\,\rho \sin\theta \sin\phi$$

$$x.\varphi = r\rho \cos(\theta - \phi)$$

didefinisikan Transformasi Fourier dari persamaan (2-5-3) :

$$\mathcal{F}_{(n)}[f(r); x_1 \to \varphi_1, x_2 \to \varphi_2, \ldots, x_n \to \varphi_n]$$

$$= (2\pi)^{-n/2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{En=-\infty}^{\infty} f(r)\, e^{i(\varphi.x)} dx$$

dimana En = menyatakan berdimensi n, jika n = 2

$$\mathcal{F}_{(2)}[f(r); x_1 \to \varphi_1, x_2 \to \varphi_2]$$

$$= \frac{1}{2\pi} \int_0^{\infty} r \int_0^{2\pi} f(r)\, e^{ir\rho\cos(\theta-\phi)} d\theta\, dr$$

karena $\frac{1}{2\pi} \int_0^{2\pi} f(r)\, e^{ir\rho\cos(\theta-\phi)} d\theta = Jo(r\rho)$

maka

$$\mathcal{F}_{(2)}[f(r); x_1 \to \varphi_1, x_2 \to \varphi_2] = \int_0^{\infty} r\, f(r)\, Jo\,(r\rho)\, dr$$

sehingga dapat didefinisikan :

$$\mathcal{F}_{(2)}\,[f(r); x_1 \to \varphi_1, x_2 \to \varphi_2] = H_o\,[f(r); \rho]$$

dengan :

$$r = (x_1^2 + x_2^2)^{1/2} \quad \text{dan} \quad \rho = (\varphi_1^2 + \varphi_2^2)^{1/2}$$

sebagai contoh jika $k > 0$

$$\mathcal{F}_{(2)}\,[r^{-1} e^{-kr}; x_1 \to \varphi_1, x_2 \to \varphi_2] = H_o\,[r^{-1} e^{-kr}; \rho]$$

dengan mengambil $v = 0$ pada persamaan (3-5-12) maka :

$$\mathcal{F}_{(2)}[r^{-1} e^{-kr}; x_1 \to \varphi_1, x_2 \to \varphi_2] = (k^2 + \rho^2)^{-1/2}, (k>0)$$

Jika diambil dalam dimensi n, maka dari definisi

Transformasi Fourier dapat diadakan substitusi :

ambil

$$\varphi_i = \rho\alpha_{ij} \quad ,(i = 1,2,3 \ldots n)$$

$$y_j = \sum_{i=1}^{n} \alpha_{ij} x_i \quad ,(j = 2,3 \ldots n)$$

$$y_1 = \sum_{i=1}^{n} \alpha_{ij} x_i$$

Koefisien α_{ij} dipilih sedemikian hingga Transformasi Orthogonal, dalam hal ini :

$$r = (y_1^2 + y_2^2 + \ldots + y_n^2)^{1/2}$$

$$dx = dy$$

$$\varphi \cdot x = \sum_{i=1}^{n} \varphi_i x_i = \rho \sum_{i=1}^{n} \alpha_{i1} x_i = \rho\, y_1$$

sekarang diambil :

$$\lambda^2 = y_2^2 + y_3^2 + \ldots + y_n^2$$

maka :

$$\mathcal{F}_{(n)} = [f(r); x_1 \to \varphi_1, \ldots x_n \to \varphi_n]$$

$$= (2\pi)^{-\frac{1}{2}n} \int_{-\infty}^{\infty} \int_{0}^{\infty} \Omega(\lambda)\, f\{(\lambda^2 + y_1^2)^{1/2}\}\, e^{i\rho y_1}\, d\lambda\, dy_1$$

dimana dapat ditulis :

$$F_{(n)}(\varphi) = \mathcal{F}_{(n)} [f(r); x_1 \to \varphi_1, \ldots, x_n \to \varphi_n]$$

dengan $\Omega(\lambda)$ didefinisikan sedemikian hingga, untuk sebarang $g(\lambda)$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{E_{n-1}=-\infty}^{\infty} g\left\{(y_2^2 + y_3^2 + \ldots + y_n^2)^{1/2}\right\} dy_2\, dy_3 \ldots dy_n$$

$$= \int_{0}^{\infty} \Omega(\lambda)\, g(\lambda)\, d\lambda$$

dimana $\Omega(\lambda)d\lambda$ menunjukan banyaknya elemen dalam En, jika diambil $\Omega(\lambda) = \omega n\ \lambda^{n-2}$ dan jika $\varrho(\lambda) = e^{-\lambda^2}$

maka :

$$\int_{-\sim}^{\sim} \int_{-\sim}^{\sim} \cdots \int_{En-1=-\sim}^{\sim} \varrho\left\{(y_2^2 + y_3^2 + \ldots + y_n^2)^{1/2}\right\} dy_2\, dy_3 \ldots dy_n$$

$$= \left\{ \int_{-\sim}^{\sim} e^{-u^2} du \right\}^{n-1} = \Pi^{1/2n - 1/2}$$

$$\text{dan } \int_0^{\sim} \Omega(\lambda)\ \varrho(\lambda)\ d\lambda = \omega n \int_0^{\sim} \lambda^{n-2}\ e^{-\lambda^2}\ d\lambda$$

$$= \frac{1}{2}\ \omega n\ \Gamma(\frac{1}{2}n - \frac{1}{2})$$

$$\frac{1}{2}\ \omega n\ \Gamma(\frac{1}{2}n - \frac{1}{2}) = \Pi^{n/2 - 1/2}$$

$$\omega n = \frac{2\pi^{n/2-1/2}}{\Gamma\ (n/2-1/2)}$$ sehingga

$$F^{\circ}_{(n)}(\varphi) = (2\pi)^{-n/2} \int_{-\sim}^{\sim} \int_0^{\sim} \frac{\lambda^{n-2}\ 2\pi^{n/2-1/2}}{\Gamma\ (n/2-1/2)} f(r)\ e^{i\rho y_1}\ d\lambda dy_1$$

$$= \frac{1}{2^{n/2-1}\ \ \Pi^{1/2}\ \Gamma(\frac{1}{2}n - \frac{1}{2})} \int_{-\sim}^{\sim} \int_0^{\sim} \lambda^{n-2} f(r)\ e^{i\rho y_1}\ d\lambda dy_1$$

dengan menggunakan Transformasi ke koordinat polar :

$\lambda = r \sin \theta$

$y_1 = r \cos \theta$

$\lambda^2 + y^2 = r^2$; $dy_1\, d\lambda = r\, dr\, d\theta$

$y_1 = -\sim \longrightarrow \sim$; $\theta = 0 \longrightarrow \Pi$

$\lambda = 0 \longrightarrow \sim$; $r = 0 \longrightarrow \sim$ maka

$$F_{(n)}(\varphi) = \frac{1}{2^{n/2-1}\ \ \Pi^{1/2}\ \Gamma(\frac{1}{2}n - \frac{1}{2})}$$

$$\int_0^{\sim} \int_0^{\pi} r^{n-2} \sin^{n-2} \theta\ f(r)\ e^{i\rho r \cos \theta}\ r\ dr\ d\theta$$

$$F_{(n)}(\varphi) = \frac{1}{2^{n/2-1}\ \Pi^{1/2}\ \Gamma(\frac{1}{2}n - \frac{1}{2})}$$

$$\int_0^{\infty} r^{n-1} f(r) \int_0^{\pi} \sin^{n-2} \theta\ e^{i\rho r \cos\theta}\, d\theta\, dr$$

karena :

$$\int_0^{\pi} \sin^{n-2} \theta\ e^{i\rho r \cos \theta}\, d\theta$$

$$= \frac{2^{n/2-1}\ \Pi^{1/2}\ \Gamma(n/2-1/2)}{(\rho r)^{n/2-1}} J_{n/2-1}(\rho r)$$

$$F_{(n)}(\varphi) = \int_0^{\infty} \rho^{1-n/2}\ r^{n/2}\ f(r)\ J_{n/2-1}(\rho r)\, dr$$

karena $F_{(n)}(\varphi)$ merupakan vaiabel ρ maka jika ditulis

$$F(\rho) = \mathcal{F}_{(n)} \left[f(r); x_1 \longrightarrow \varphi_1, x_2 \longrightarrow \varphi_2, \ldots, x_n \longrightarrow \varphi_n\right]$$

$$\rho^{n/2-1} F(\rho) = \int_0^{\infty} r^{1/2n} f(r)\ J_{1/2n-1}(\rho r)\, dr$$

ambil $v = n/2-1$

$$\rho^{v}\ F(\rho) = \int_0^{\infty} r^{v+1} f(r)\ J_v(\rho r)\, dr$$

$$\rho^{v}\ F(\rho) = H_v\left[r^{v} f(r); \rho\right] \qquad (3\text{-}1\text{-}1)$$

dengan menggunakan Invers Transformasi Hankel yang akan dibicarakan pada sub bab berikutnya, akan didapat :

$$r^{v}\ f(r) = H_v\ [\ \rho^{v}\ F(\rho); r]$$

$$f(r) = r^{-v}\ H_v\ [\ \rho^{v}\ F(\rho); r]$$

misalkan $r^{v} f(r) = \phi(r)$ dan $\rho^{v} F(\rho) = \bar{\phi}_v(\rho)$ maka :

$$\bar{\phi}_v(\rho) = H_v\ [\phi(r); \rho] \qquad (3\text{-}1\text{-}3)$$

$$\phi(r) = H_v\ [\bar{\phi}_v(\rho); r] \qquad (3\text{-}1\text{-}4)$$

jika $\bar{\phi}v(\rho)$ Transformasi Hankel order v dari sebarang fungsi $\phi(r)$ maka $\phi(r)$ adalah Transformasi Hankel order v dari $\bar{\phi}v(\rho)$. Hasil ini disebut Theorema Hankel Invers.

3.2. Sifat-sifat Transformasi Hankel

Didefinisikan :

$$Hv \left[f(ax) ; \varphi \right] = \int_0^\infty x\ f(ax)\ Jv\ (x\varphi)\ dx \qquad (a>0)$$

jika ρ = ax akan didapatkan :

$$Hv\ [\ f(\rho)\ ;\ \varphi\ ] = \frac{1}{a^2} \int_0^\infty x\ f(\rho)\ Jv\ (\rho\varphi/a)\ d\rho$$

dapat ditulis :

$$Hv \left[f(\rho);\varphi \right] = a^{-2}\ Hv\ \left[f(\rho);\varphi/a\right] \qquad (3\text{-}2\text{-}1)$$

Didefinisikan :

$$Hv\ [f(x);\ H(x-a);\varphi] = \int_0^\infty x\ f(x)\ Jv(\varphi x)\ H(x-a)dx$$

$$= \int_a^\infty x\ f(x)\ Jv(\varphi x)\ dx \qquad (3\text{-}2\text{-}2)$$

demikian juga :

$$Hv\ [f(x);\ H(a-x);\varphi] = \int_0^\infty x\ f(x)\ Jv(\varphi x)\ H(a-x)dx$$

$$= \int_0^a x\ f(x)\ Jv(\varphi x)\ dx \qquad (3\text{-}2\text{-}3)$$

dimana H dinamakan "Heaviside unit function"

Pandang fungsi $|\ X^{1/2}\ Jv(x)\ |$ terbatas untuk x real (3) dapat juga dinyatakan :

$$|\ X^{1/2}\ Jv(x)\ | \le A,\ x\ \text{real},\ v \ge -\tfrac{1}{2} \qquad (3\text{-}2\text{-}4)$$

dimana A bilangan positif

$$| \mathcal{F}v(\varphi) | = \left| \int_0^\infty x \, f(x) \, Jv(\varphi x) \, dx \right|$$

$$= \left| \int_0^\infty x^{1/2} \, f(x) \, x^{1/2} \, Jv(\varphi x) \, dx \right|$$

$$\leq \left| \varphi^{1/2} \int_0^\infty \left| x^{1/2} \, f(x) \right| \left| (x\varphi)^{1/2} \, Jv(\varphi x) \right| dx \right.$$

$$\leq A \, \varphi^{1/2} \int_0^\infty \left| x^{1/2} \, f(x) \right| dx$$

Karena $\sqrt{x} \, f(x)$ absolut integrabel untuk bilangan real positif maka $\mathcal{F}v(\varphi)$ terbatas untuk semua φ real.

3.3. Invers Transformasi Hankel

Untuk membuktikan teorema Invers Transformasi Hankel order v, tidak lepas dari sifat-sifat fungsi Bessel. Dalam penggunaan pada umumnya hanya Transformasi Hankel order nol dan satu untuk detailnya digunakan sifat-sifat dari fungsi Bessel pandang fungsi :

$\int_0^x t^{1/2} Jv(t)dt$ terbatas untuk semua nilai $x \geq 0$

Jika $v \geq -\frac{1}{2}$

maka dapat juga ditulis :

$$\left| \int_0^x t^{1/2} Jv(t)dt \right| \leq B \quad , \; v \geq -\frac{1}{2} \qquad (3\text{-}3\text{-}1)$$

dari sini didapat hubungan :

$$\int_{\alpha}^{\beta} t^{1/2} Jv(\lambda x)dx = \lambda^{-3/2} \int_{\alpha\lambda}^{\beta\lambda} t^{1/2} Jv(t)dt$$

dengan menggunakan subst. : $\lambda x = t$

dapat disimpulkan $\beta > \alpha \geq 0, \quad \lambda > 0$

$$\left| \int_{\alpha}^{\beta} x^{1/2} Jv(\lambda x)dx \right| \leq 2B\ \lambda^{-3/2} \qquad (3\text{-}3\text{-}2)$$

hubungan serupa didapat dari pers. (3-2-4)

sebab jika $\beta \geq \alpha \geq 0$, $\lambda > 0$ dapat ditulis :

$$\int_{\alpha}^{\beta} W(x)\ x^{1/2} Jv(\lambda x)dx = \lambda^{-1/2} \int_{\alpha}^{\beta} W(x)(\lambda x)^{1/2} Jv(\lambda x)dx$$

sehingga didapat :

$$\left| \int_{\alpha}^{\beta} W(x)\ x^{1/2} Jv(\lambda x)dx \right| \leq A\ \lambda^{-1/2} \int_{\alpha}^{\beta} |W(x)|\ dx \qquad (3\text{-}3\text{-}3)$$

pandang differensial fungsi Bessel.

$$\frac{\partial}{\partial u} \left[xU^{v+1} Jv{+}1(xv)\ U^{-v} Jv(yu) - yu^{-v} Jv(xu) U^{v+1} Jv{+}1(yu) \right]$$

$$* \ \frac{\partial}{\partial u} \left[xU^{v+1} Jv{+}1(xu) \right] = x^2\ U^{v+1} Jv(ux)$$

$$* \ \frac{\partial}{\partial u} \left[yU^{v+1} Jv{+}1(yv) \right] = y^2\ U^{v+1} Jv\ (uy)$$

$$* \ \frac{\partial}{\partial u} \left[U^{-v} Jv(yu) \right] = -y\ U^{-v} Jv{+}1(uy)$$

$$* \ \frac{\partial}{\partial u} \left[U^{-v} Jv(xu) \right] = -x\ U^{-v} Jv{+}1(ux)$$

sehingga :

$$= x^2 UJv(ux)Jv(uy) - xyuJv{+}1(ux)Jv{+}1(uy) + xyuJv{+}1(ux)Jv{+}1(uy)$$

$$-y^2 UJv(xu)Jv(uy)$$

$$= (x^2-y^2)\ uJv(xu)Jv(yu) \quad \text{maka}$$

$$\frac{\partial}{\partial u}\left[xU^{v+1}Jv_{+1}(xv)\ U^{-v}Jv(yu)-yu^{-v}Jv(xu)U^{v+1}Jv_{+1}(yu)\right] =$$

$$(x^2-y^2)\ uJv(xu)Jv(yu) \qquad (3\text{-}3\text{-}4)$$

Jika $v > -\frac{1}{2}$ maka

$$\int_0^{\lambda}\frac{\partial}{\partial u}\left[\frac{xU^{v+1}Jv_{+1}(xv)\ U^{-v}Jv(yu)-yu^{-v}Jv(xu)U^{v+1}Jv_{+1}(yu)}{(x^2-y^2)}\right]du$$

$$= \int_0^{\lambda} u\ Jv(xu)Jv(yu)du$$

$$\int_0^{\lambda} u\ Jv(xu)Jv(yu)du = hv(x,y;\lambda) \quad ,v > -\frac{1}{2} \qquad (3\text{-}3\text{-}5)$$

dengan :

$$hv(x,y;\lambda) = \frac{\lambda xJv_{+1}(\lambda x)Jv(\lambda y)-\lambda yJv(\lambda x)Jv_{+1}(\lambda y)}{(x^2-y^2)} \qquad (3\text{-}3\text{-}6)$$

Untuk membuktikan lebih lanjut theorema invers transformasi Hankel terlebih dahulu dikaji Lemma-lemma berikut :

Lemma I (The Riemann - Lebesque Lemma)

Jika $x^{1/2}f(x)$ kontinue sepotong-sepotong dan dapat diintegralkan secara mutlak pada (a,b), ($b \geq a \geq 0$) dengan $\lambda \longrightarrow \sim$

$$\int_a^b x\ f(x)Jv(\lambda x)dx = O(\lambda^{-1/2}),\ v > -\frac{1}{2} \qquad (3\text{-}3\text{-}7)$$

Bukti :

Jika b finite maka integral (a,b) dapat dibagi dalam n sub interval (x_{s-1}, x_s) $(s = 1,2,\ldots.n)$ dengan $x_0 = a$, $x_n = b$ untuk setiap $\delta > 0$ dapat dipilih n cukup besar sehingga

$$\sum_{s=1}^{n} (M_s - m_s)\delta < \varepsilon \qquad (3\text{-}3\text{-}8)$$

$$\delta = x_s - x_{s-1} \qquad (s = 1,2,\ldots..n)$$

dimana $m_s \leq x^{1/2} f(x) \leq M_s$ untuk $\forall x \varepsilon (x_{s-1}, x_s)$ bagaimanapun juga dapat dinyatakan :

$$\sqrt{x}\, f(x) = \sqrt{x_{s-1}}\, f(x_{s-1}) + W_s(x) \quad , \forall x \varepsilon (x_{s-1}, x_s) \qquad (3\text{-}3\text{-}9)$$

dengan $|W_s(x)| \leq M_s - m_s$ (3-3-10)

dan $\int_a^b x\ f(x) Jv(\lambda x) dx = I_1 + I_2$ (3-3-11)

dimana I_1 dan I_2 didefinisikan sebagai berikut :

$$I_1 = \sum_{s=1}^{n} \sqrt{x_{s-1}}\, f(x_{s-1}) \int_{x_{s-1}}^{x_s} x^{1/2} Jv(\lambda x) dx \qquad (3\text{-}3\text{-}12)$$

$$I_2 = \sum_{s=1}^{n} \int_{x_{s-1}}^{x_s} x^{1/2} W_s(x) Jv(\lambda x) dx \qquad (3\text{-}3\text{-}13)$$

berturut-turut berdasarkan pers (3-3-2) dapat dinyatakan :

$$|I_1| \leq 2B\ \lambda^{-3/2} \sum_{s=1}^{n} \left| \sqrt{x_{s-1}}\, f(x_{s-1}) \right| \leq 2nBM\ \lambda^{-3/2}$$

dimana M = Batas atas terkecil $\sqrt{x}\, f(x)$

juga dari pers (3-3-3) dapat ditemukan :

$$|I_2| \leq \lambda^{-1/2} A \sum_{s=1}^{n} \int_{x_{s-1}}^{x_s} |\ W_s(x)\ |\ dx$$

$$\leq A\ \lambda^{-1/2} \sum_{s=1}^{n} \int_{x_{s-1}}^{x_s} (M_s - m_s)\,dx$$

$$= A\ \lambda^{-1/2} \sum_{s=1}^{n} (M_s - m_s)(x_s - x_{s-1})$$

$$\leq A\ \lambda^{-1/2}\ \varepsilon$$

$|I_2| < \lambda^{-1/2} A\varepsilon$ sehingga

$$\left| \int_a^b x\ f(x) J_v(\lambda x)\,dx \right| < 2\ n\ B\ M\ \lambda^{-3/2} + \lambda^{-1/2}\ A\ \varepsilon$$

untuk ε dapat ditemukan n; untuk nilai n ini dapat dipilih λ cukup besar sedemikian hingga :

$$\left| \int_a^b x\ f(x) J_v(\lambda x)\,dx \right| < 2\ \lambda^{-1/2}\ A\ \varepsilon \qquad (3\text{-}3\text{-}14)$$

Jika hal lain b infinite maka dapat dinyatakan :

$$\left| \int_a^{\infty} x\ f(x) J_v(\lambda x)\,dx \right| \leq \left| \int_a^b x\ f(x) J_v(\lambda x)\,dx \right|$$

$$+ \left| \int_b^{\infty} x\ f(x) J_v(\lambda x)\,dx \right|$$

$$\leq \left| \int_a^b x\ f(x) J_v(\lambda x)\,dx \right| + \lambda^{-1/2}\ A \int_a^{\infty} x^{-1/2} |f(x)|\,dx$$

dapat dipilih b cukup besar sehingga :

$$\int_a^{\infty} x^{-1/2} |f(x)|\,dx < \varepsilon$$

$$\left| \int_a^b x\ f(x) J_v(\lambda x)\,dx \right| < 2\ \lambda^{-1/2}\ A\ \varepsilon + \lambda^{-1/2}\ A\ \varepsilon$$

$= 3\ \lambda^{-1/2}\ A\ \varepsilon$ terbukti.

Lemma 2

Jika $x^{1/2}f(x)$ kontinue sepotong-sepotong dan dapat diintegralkan secara mutlak pada $(0, \sim)$ dan $\bar{f}v(u) = Hv[f(\rho);\ u]$ maka $\int_0^{\sim} u\ \bar{f}v(u)Jv(ru)du$

$$= \lim_{\lambda\to\sim} \int_0^{\sim} \rho\ f(\rho)\ hv(\rho,r;\ \lambda)\ d\rho, \quad v > -\frac{1}{2}$$

Bukti :

Dibuktikan bahwa limit sisi kanan ada, maka dapat disederhanakan sedemikian hingga :

$$F(u,\rho;\ r) = \rho u\ f(\rho)Jv(\rho u)Jv(ru)$$

dari pers (3-2-4) didapatkan :

$$|\ x^{1/2}Jv(x)\ | \le A$$

$$|u^{1/2}Jv(\rho u)| = \rho^{-1/2}\ |(u\rho)^{1/2}Jv(\rho u)| \le \rho^{-1/2}\ A$$

$$|\ u^{1/2}\ Jv(\rho u)| = r^{-1/2}\ |(ur)^{1/2}\ Jv(ru)| \le r^{-1/2}\ A$$

$$|\rho u\ f(\rho)Jv(\rho u)Jv(ru)|$$

$$= \ |\rho\ \ f(\rho)\ \ u^{1/2}\ \ Jv(\rho u)u^{1/2}\ Jv(ru)|$$

$$\le A^2\rho^{1/2}\ r^{-1/2}\ \ |f(\rho)| \tag{3-3-15}$$

maka

$$\left|\ \int_0^{\sim}\int_0^{\lambda} F(u,\rho;r)du d\rho - \int_0^{\lambda}\int_0^{\sim} F(u,\rho;r)d\rho du\ \right| \le F_1 + F_2$$

dimana

$$F_1 = \int_c^{\sim}\int_0^{\lambda} |F(u,\rho;r)|\ du d\rho$$

$$F_2 = \int_0^{\lambda}\int_c^{\sim} |F(u,\rho;r)|\ d\rho du$$

dimana $c > 0$, hasil ini benar untuk sembarang c tetapi jika diberikan sebarang bilangan positif ε, dapat dipilih c sedemikian hingga

$$\int_c^{\sim} x^{1/2} \left|F(x)\right| dx < \frac{\varepsilon\, r^{1/2}}{2\, A^2\, \lambda} \qquad (3\text{-}3\text{-}16)$$

dimana A konstanta

Karena $\left| F(U,\rho;r)\right| \leq A^2 \rho^{1/2} r^{-1/2} \left|f(\rho)\right|$ maka

$$F_1 = \int_0^{\lambda}\int_c^{\sim} \left|F(U,\rho;r)\right| dU d\rho$$

$$\leq \int_c^{\sim}\int_0^{\lambda} A^2 \rho^{1/2} r^{-1/2} \left|f(\rho)\right| dU d\rho$$

$$= \int_c^{\sim} \lambda A^2 \rho^{1/2} r^{-1/2} \left|f(\rho)\right| d\rho$$

$$\leq A^2 \lambda r^{-1/2} \int_c^{\sim} \rho^{1/2} \left|f(\rho)\right| d\rho < \lambda A^2 r^{-1/2} \frac{\varepsilon\, r^{1/2}}{2\, A^2\, \lambda}$$

$$= \frac{\varepsilon}{2}$$

Demikian juga untuk F_2

$$F_2 = \int_0^{\lambda}\int_c^{\sim} F(U,\rho;r) d\rho dU$$

$$\leq \int_0^{\lambda}\int_c^{\sim} A^2 \rho^{1/2} r^{-1/2} \left|f(\rho)\right| d\rho dU$$

$$= \int_0^{\lambda} A^2 r^{-1/2} \int_c^{\sim} \rho^{1/2} \left|f(\rho)\right| d\rho dU$$

$$< A^2 r^{-1/2} \int_0^{\lambda} \frac{\varepsilon\, r^{1/2}}{2\, A^2\, \lambda} du < \frac{\varepsilon}{2\lambda} \int_0^{\lambda} dU = \frac{\varepsilon}{2}$$

dengan demikian

$$0 \leq \left| \int_0^{\sim}\int_0^{\lambda} F(U,\rho;r) dU d\rho - \int_0^{\lambda}\int_0^{\sim} F(U,\rho;r) d\rho dU \right| < \varepsilon$$

maka

$$\int_0^{\infty}\int_0^{\lambda} F(\upsilon,\rho;r)\,d\upsilon\, d\rho = \int_0^{\lambda}\int_0^{\infty} F(\upsilon,\rho;r)\,d\rho\, d\upsilon$$

maka

$$\int_0^{\infty}\int_0^{\infty} \rho\upsilon\, f(\rho) J_v(\rho\upsilon) J_v(r\upsilon)\,d\rho\, d\upsilon$$

$$= \lim_{\lambda\to\infty} \int_0^{\infty}\int_0^{\lambda} \rho\upsilon\, f(\rho) J_v(\rho\upsilon) J_v(r\upsilon)\,d\upsilon\, d\rho$$

$$\int_0^{\infty} \upsilon\, J_v(r\upsilon) \int_0^{\infty} \rho\, f(\rho) J_v(\rho\upsilon)\,d\rho\, d\upsilon$$

$$= \lim_{\lambda\to\infty} \int_0^{\infty} \rho f(\rho) \int_0^{\lambda} \omega J_v(\rho\upsilon) J_v(r\upsilon)\,d\upsilon\, d\rho$$

Kemudian dari (3-3-5)

$$\int_0^{\lambda} \omega J_v(\rho\upsilon)\, \bar{f}_v(\upsilon)\,d\upsilon = \lim_{\lambda\to\infty} \int_0^{\infty} \rho\, f(\rho)\ h_v(\rho,r;\lambda)\,d\rho$$

terbukti

<u>**Lemma 3**</u>

Jika $x^{1/2} f(x)$ kontinue sepotong-sepotong dan dapat diintegralkan secara mutlak pada garis real positif dan r tidak termuat dalam interval (a,b) maka

$$\int_0^{\infty} \omega J_v(r\upsilon) \int_a^b \rho\, f(\rho) J_v(\rho\upsilon)\,d\rho\, d\upsilon = 0 \quad , \ v > -\tfrac{1}{2}$$

Bukti :

Dengan menggunakan lemma 2 didapat :

$$\int_0^{\infty} \omega J_v(r\upsilon) \int_a^b \rho\, f(\rho) J_v(\rho\upsilon)\,d\rho\, d\upsilon = \lim_{\lambda\to\infty} \int_a^b \rho\, f(\rho)\ h_v(\rho,r;\lambda)\,d\rho$$

dimana :

$$h_v(\rho,r;\lambda) = \frac{\lambda\rho J_{v+1}(\lambda\rho)J_v(\lambda r)-\lambda r J_v(\lambda\rho)J_{v+1}(\lambda r)}{\rho^2 - r^2}$$

pandang :

$$\lim_{\lambda\to\infty} \int_a^b \rho\, f(\rho)\; h_v(\rho,r;\lambda)d\rho = f_1 - f_2$$

dimana

$$f_1 = \lim_{\lambda\to\infty} J_v(\lambda r) \int_a^b \frac{\lambda\rho^2 f(\rho)}{\rho^2-r^2} J_{v+1}\ (\lambda\rho)d\rho$$

$$f_2 = \lim_{\lambda\to\infty} r\ J_{v+1}(\lambda r) \int_a^b \frac{\lambda\rho\ f(\rho)}{\rho^2-r^2} J_v\ (\lambda\rho)d\rho$$

Sebab

$$f_1 - f_2 = \lim_{\lambda\to\infty} \left[J_v(\lambda r) \int_a^b \frac{\lambda\rho^2 f(\rho)}{\rho^2-r^2} J_{v+1}\ (\lambda\rho)d\rho - r\ J_{v+1}(\lambda r) \int_a^b \frac{\lambda\rho\ f(\rho)}{\rho^2-r^2} J_v\ (\lambda\rho)d\rho \right]$$

$$= \lim_{\lambda\to\infty} \left[\int_a^b \rho\ f(\rho) \left\{ \frac{\lambda\rho J_{v+1}(\lambda\rho)J_v(\lambda r)-\lambda r J_v(\lambda\rho)J_{v+1}(\lambda r)}{\rho^2 - r^2} \right\} d\rho \right]$$

$$= \lim_{\lambda\to\infty} \int_a^b \rho\ f(\rho)\ h_v(\rho,r;\lambda)d\rho$$

Dengan menggunakan Lemma 1 (Lemma Riemann - Lebesque), maka :

$$f_1 = \lim_{\lambda\to\infty} J_v(\lambda r) \int_a^b \frac{\lambda\rho^2 f(\rho)}{\rho^2-r^2} J_{v+1}\ (\lambda\rho)d\rho$$

$$= \lim_{\lambda\to\infty} \lambda J_v(\lambda r) \lim_{\lambda\to\infty} \int_a^b \frac{\rho^2 f(\rho)}{\rho^2-r^2} J_{v+1}\ (\lambda\rho)d\rho = 0$$

Secara analog maka :

$$f_2 = \lim_{\lambda \to \sim} r\, J_{v+1}(\lambda r) \int_a^b \frac{\lambda\rho\, f(\rho) J_v(\lambda\rho)\, d\rho}{\rho^2 - r^2}\, J_v\, (\lambda\rho)\, d\rho = 0$$

sehingga terbukti :

$$\int_0^{\sim} U J_v(rU) \int_a^b \rho\, f(\rho) J_v(\rho U)\, d\rho\, dU = 0$$

<u>Lemma 4</u>

Jika $x(\rho) = \rho^{1/2} f(\rho)$ kontinue sepotong-sepotong dan dapat diintegralkan secara mutlak pada bilangan real positif dan dipilih δ, sedemikian hingga untuk setiap $\varepsilon > 0$ berlaku $|x(\rho) - x(r+)| < \varepsilon$, $r < \rho < r+\delta$ dengan $\lambda \longrightarrow \sim$

maka :

$$\int_0^{\lambda} U J_v(Ur) \int_r^{r+\delta} \rho\, f(\rho) J_v(U\rho)\, d\rho\, dU = \tfrac{1}{2} f(r+), \quad v > -\tfrac{1}{2}$$

(3-3-17)

demikian juga jika $|x(\rho) - x(r-)| < \varepsilon$, $r-\delta < \rho < r$ dengan $\lambda \longrightarrow \sim$ maka :

$$\int_0^{\lambda} U J_v(Ur) \int_{r-\delta}^{r} \rho\, f(\rho) J_v(U\rho)\, d\rho\, dU = \tfrac{1}{2} f(r-), \quad v > -\tfrac{1}{2}$$

(3-3-18)

Bukti :

$$\int_0^{\lambda} U\, Jv(Ur) \int_r^{r+\delta} \rho\, f(\rho) Jv(U\rho)\, d\rho\, dU$$

$$= \int_r^{r+\delta} \rho^{1/2} \rho^{1/2}\, f(\rho) \int_0^{\lambda} U\, Jv(Ur) Jv(U\rho)\, dU\, d\rho$$

$$= \int_r^{r+\delta} \rho^{1/2} x(\rho) \int_0^{\lambda} U\, Jv(Ur) Jv(U\rho)\, dU\, d\rho$$

$$= \int_r^{r+\delta} \rho^{1/2} r^{1/2}\, f(r+) \int_0^{\lambda} U\, Jv(Ur) Jv(U\rho)\, dU\, d\rho$$

$$= f(r+) \int_r^{r+\delta} (\rho r)^{1/2} \int_0^{\lambda} U\, Jv(Ur) Jv(U\rho)\, dU\, d\rho$$

Karena :

$$Jv(\rho, r; p) = \int_0^{\infty} u e^{-p^2 u^2}\, Jv(ru) Jv(\rho u)\, du$$

$$= \frac{1}{2p^2}\, Iv\left[\frac{r\rho}{2p^2} \right] \exp\left[- \frac{r^2+\rho^2}{4p^2} \right]$$

dimana $Iv(x) = i^{-v}\, Jv(ix)$

untuk $p \longrightarrow 0$

$$Iv\left[\frac{r\rho}{2p^2} \right] = \frac{p}{(\pi\rho r)^{1/2}} \exp\left[\frac{r\rho}{2p^2} \right]$$

sehingga

$$Jv(\rho,r;p) = \frac{1}{2p\ (\pi r\rho)^{1/2}} \exp\left(-\frac{(r-\rho)^2}{4p^2}\right)$$

oleh sebab itu :

$$\int_r^{r+\delta} (\rho r)^{1/2}\ Jv(\rho,r;p)d\rho$$

$$= \frac{1}{2p\ (\pi)^{1/2}} \int_r^{r+\delta} \exp\left(-\frac{(r-\rho)^2}{4p^2}\right)d\rho$$

substitusi $\frac{r - \rho}{2p} = -z \longrightarrow d\rho = 2pdz$

jika $\rho = r$ maka $z = 0$

$\rho = r + \delta$ maka $z = \delta/2p$

$$\int_r^{r+\delta} (\rho r)^{1/2}\ Jv(\rho,r;p)d\rho = \left(\frac{1}{\pi}\right)^{1/2} \int_0^{\delta/2p} e^{-z^2} dz$$

untuk $\delta > 0$

$$\lim_{p\to 0} \int_r^{r+\delta} (\rho r)^{1/2}\ Jv(\rho,r;p)d\rho = \tfrac{1}{2}$$

$$\lim_{p\to 0} f(r+) \int_r^{r+\delta} (\rho r)^{1/2}\ Jv(\rho,r;p)d\rho = \tfrac{1}{2}\ f(r+)$$

sehingga dengan melihat definisi $Jv(\rho,r;p)$ maka

$$f(r+) \int_r^{r+\delta} (\rho r)^{1/2} \int_0^{\lambda} uJv(ru)Jv(\rho u)dud\rho = \tfrac{1}{2}\ f(r+)$$

dengan demikian

$$\int_0^\lambda uJ_v(ur) \int_r^{r+\delta} \rho f(\rho)\; J_v(\rho u)d\rho du = \frac{1}{2}\, f(r+),$$

$$v > -\tfrac{1}{2},\ \lambda > \infty$$

dengan cara yang sama akan didapat :

$$\int_0^\lambda uJ_v(ur) \int_{r-\delta}^{r} \rho f(\rho)\; J_v(\rho u)d\rho du = \frac{1}{2}\, f(r-),$$

$$v > -\tfrac{1}{2},\ \lambda > \infty$$

Theorema I (The Hankel Invers Theorema)

Jika $x^{1/2} f(x)$ kontinue sepotong-sepotong dan dapat diintegralkan secara mutlak pada bilangan positif maka jika :

$\bar{f}_v(U) = H_v[f(\rho);U]$, $v > -\frac{1}{2}$ ada maka

$$\int_0^\infty U\, \bar{f}_v(U)\; J_v(Ur)dU = \frac{1}{2}[f(r+) + f(r-)]$$

Bukti :

Dari lemma 3 dan 4, jika dinyatakan :

$$I(\alpha,\beta) = \lim_{\lambda\to\infty} \int_0^\lambda U\, J_v(Ur) \int_\alpha^\beta \rho\; f(\rho)J_v(U\rho)d\rho dU$$

dari lemma 3, dimana r tidak didalam $(0,r-\delta)$, $(r+\delta, \infty)$ maka akan didapat

$$I(0,r-\delta) = \lim_{\lambda\to\infty} \int_0^\lambda U\, J_v(Ur) \int_0^{r-\delta} \rho\; f(\rho)J_v(U\rho)d\rho dU = 0$$

$$I(r+\delta,\sim) = \lim_{\lambda\to\sim} \int_0^{\lambda} U\, Jv(Ur) \int_{r+\delta}^{\sim} \rho\, f(\rho) Jv(U\rho)\, d\rho\, dU = 0$$

Demikian juga jika digunakan Lemma A

$$I(r-\delta,r) = \lim_{\lambda\to\sim} \int_0^{\lambda} U\, Jv(Ur) \int_{r-\delta}^{r} \rho\, f(\rho) Jv(U\rho)\, d\rho\, dU = \frac{1}{2}\, f(r-)$$

$$I(r,r+\delta) = \lim_{\lambda\to\sim} \int_0^{\lambda} U\, Jv(Ur) \int_{r}^{r+\delta} \rho\, f(\rho) Jv(U\rho)\, d\rho\, dU = \frac{1}{2}\, f(r+)$$

Karena :

$$\int_0^{\sim} U\, \bar{f}v(U)\, Jv(Ur)\, dU = \lim_{\lambda\to\sim} \int_0^{\lambda} U\, Jv(Ur) \int_0^{\sim} \rho\, f(\rho) Jv(U\rho)\, d\rho\, dU$$

$$= \lim_{\lambda\to\sim} \int_0^{\lambda} U\, Jv(Ur) \left[\int_0^{r-\delta} \rho\, f(\rho) Jv(U\rho)\, d\rho + \int_{r-\delta}^{r} \rho\, f(\rho) Jv(U\rho)\, d\rho \right.$$

$$\left. + \int_{r}^{r+\delta} \rho\, f(\rho) Jv(U\rho)\, d\rho + \int_0^{\sim} \rho\, f(\rho) Jv(U\rho)\, d\rho \right] du$$

$$= I(0,r-\delta) + I(r-\delta,r) + I(r,r+\delta) + I(r+\delta,\sim)$$

$$= \frac{1}{2}\, [f(r+) + f(r-)]$$

Sehingga

$$\int_0^{\sim} U\, \bar{f}v(U)\, Jv(Ur)\, dU = \frac{1}{2}\, [f(r+) + f(r-)]$$

Jika $f(\rho)$ kontinue di titik $\rho = r$ maka

$$\int_0^{\sim} U\, \bar{f}v(U)\, Jv(Ur)\, dU = f(r)$$

$$Hv[\bar{f}v(U);r] = f(r)$$

Dari definisi didapat :

$$\bar{f}v(\varphi) = Hv[f(x);\varphi]$$

Fungsi $\overline{f}v(\varphi)$ disebut Transformasi Hankel order v dari fungsi $f(x)$.

Sedangkan $f(x)$ sendiri dapat dinyatakan dengan :

$$f(x) = Hv[\overline{f}_v(\varphi);\ x]$$

dan ini sering dinyatakan, bahwa $f(x)$ adalah invers transformasi Hankel dari $\overline{f}v(\varphi)$ atau dapat ditulis :

$$Hv = Hv^{-1} \quad , v > -\tfrac{1}{2}$$

3.4. Fungsi Derivatif dari Transformasi Hankel

Didalam penerapan transformasi Hankel pada masalah fisika dan didalam perhitungan transformasi Hankel dari fungsi elementer, tidak terlepas dari derivatif transformasi Hankel. Dengan menggunakan definisi transformasi Hankel maka didapat :

$$Hv\left[\rho^{v-1}\ \frac{\partial}{\partial\rho}\left\{\rho^{1-v}f(\rho)\right\};\ \varphi\right] = \int_0^\infty \rho^{v-1}\rho Jv(\varphi\rho)\ \frac{\partial}{\partial\rho}\left\{\rho^{1-v}f(\rho)\right\} d\rho$$

$$= \int_0^\infty \rho^{v} Jv(\varphi\rho)\ \frac{\partial}{\partial\rho}\left\{\rho^{1-v}f(\rho)\right\} d\rho$$

$$= \Big[\ \rho f(\rho) Jv(\varphi\rho)\Big]_0^\infty \int_0^\infty \rho^{v-1}f(\rho)\ \frac{\partial}{\partial\rho}\left\{\rho^{v} Jv(\varphi\rho)\right\} d\rho$$

Suku pertama sama dengan nol, dilengkapi bahwa $f(\rho)$ sedemikian hingga :

$$\lim_{\rho\to 0} \rho^{1} f(\rho) = 0 \qquad\qquad \lim_{\rho\to\infty} \rho^{1/2} f(\rho) = 0$$

atau dapat pula ditulis :

$$f(\rho) = O(\rho^{-v-1}) \quad ,\rho \longrightarrow 0 \qquad\qquad (3\text{-}4\text{-}1)$$

Karena :

$$\frac{\partial}{\partial\rho}\left\{\rho^{v}Jv(\varphi\rho)\right\} d\rho = \varphi\rho^{v}Jv_{-1}(\varphi\rho)$$

Dengan demikian $Hv\left[\rho^{v-1}\ \frac{\partial}{\partial\rho}\left\{\rho^{1-v}f(\rho)\right\};\ \varphi\right]$

$$= -\varphi\int_{0}^{\sim}\rho\ f(\rho)\ Jv_{-1}(\varphi\rho)\ d\rho$$

maka, dilengkapi $f(\rho)$ memenuhi kondisi (3-4-1) sedemikian hingga transformasi Hankel order v dan v_{-1} ada. didapat hubungan :

$$Hv\left[\rho^{v-1}\ \frac{\partial}{\partial\rho}\left\{\rho^{1-v}f(\rho)\right\};\ \varphi\right] = -\varphi\ Hv_{-1}\left[\ f(\rho);\ \varphi\right] \qquad (3\text{-}4\text{-}2)$$

Jika diambil v = 1 akan didapat :

$$H_{1}\left[\rho^{0}\ \frac{\partial}{\partial\rho}\left\{\rho^{0}f(\rho)\right\};\ \varphi\right] = -\varphi\ Ho\left[\ f(\rho);\ \varphi\right] \qquad (3\text{-}4\text{-}3)$$

$$H_{1}\left[\ f(\rho);\ \varphi\right] = -\varphi\ Ho\left[\ f(\rho);\ \varphi\right]$$

dimana :

$$f(\rho) = O(\rho^{-1}) \quad ,\rho \longrightarrow 0$$

dengan jalan yang sama :

$$Hv\left[\rho^{-v-1}\ \frac{\partial}{\partial\rho}\left\{\rho^{v+1}f(\rho)\right\};\ \varphi\right]$$

$$= \int_{0}^{\sim}\rho^{-v-1}Jv(\varphi\rho)\ \frac{\partial}{\partial\rho}\left\{\rho^{v+1}f(\rho)\right\} d\rho$$

$$= \int_{0}^{\sim}\rho^{-v}Jv(\varphi\rho)\ d\left\{\rho^{v+1}\ f(\rho)\right\}$$

$$= \rho Jv(\varphi\rho)\ f(\rho)\Big]_{0}^{\sim} - \int_{0}^{\sim}\rho^{v+1}f(\varphi\rho)\ \frac{\partial}{\partial\rho}\left\{\rho^{-v}Jv(\rho\varphi)\right\} d\rho$$

Suku pertama nol; jika $f(\rho)$ memenuhi kondisi yang sama dengan yang di atas, dan jika digunakan rumus recurensi :

$$\frac{\partial}{\partial\rho}\left\{\rho^{-v}Jv(\rho\varphi)\right\}d\rho = -\varphi\ \rho^{-v}Jv{+1}(\rho\varphi)$$

maka :

$$Hv\left[\rho^{-v-1}\ \frac{\partial}{\partial\rho}\left\{\rho^{v+1}f(\rho)\right\};\ \varphi\right] = -\varphi\int_0^{\infty}\rho\ f(\rho)\ Jv{+1}(\varphi\rho)\ d\rho$$

de ngan jalan yang sama jika $f(\rho)$ memenuhi persamaan (3-4-1) :

$$Hv\left[\rho^{-v-1}\ \frac{\partial}{\partial\rho}\left\{\rho^{v+1}f(\rho)\right\};\ \varphi\right] = \varphi\ Hv{+1}\left[f(\rho);\ \varphi\right] \qquad (3\text{-}3\text{-}4)$$

dalam hal khusus, diambil $v = 0$, maka

$$Ho\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left\{\rho^{v+1}f(\rho)\right\};\ \varphi\right] = \varphi\ H_1\left[f(\rho);\ \varphi\right] \qquad (3\text{-}3\text{-}5)$$

karena :

$$f(\rho) = O(\rho^{-1}) \quad ,\rho \longrightarrow 0$$

Didefinisikan operator differensial :

$$Bv = \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\ \frac{\partial}{\partial\rho} - \frac{v^2}{\rho^2} \qquad (3\text{-}4\text{-}5')$$

dari definisi transformasi Hankel order nol didapat

$$Ho\ [Bo\ f;\ \varphi] = Ho\left[\frac{1}{\rho}\ \frac{\partial}{\partial\rho}\left\{\rho\ \frac{\partial f}{\partial\rho}\right\};\ \varphi\right]$$

dengan menggunakan persamaan (3-4-5) dengan :

$$\frac{\partial f}{\partial\rho} = O(\rho^{-1}),\ \rho \longrightarrow 0 \qquad \text{maka}$$

$$Ho\ [Bo\ f;\ \varphi] = -\varphi\ H_1\left[\frac{\partial f}{\partial\rho}\ ;\ \varphi\right]$$

Sekarang jika :

$$f(\rho) = O(\rho^{-1}),\ \rho \longrightarrow 0$$

dari persamaan (3-4-3), maka

$$Ho\,[Bo\ f;\ \varphi] = -\varphi^2\ Ho\left[\ f\ ;\ \varphi\right]$$

persamaan (3-4-5') dapat dibuat secara umum, dalam hal ini $v \neq 0$

$$Bvf = \rho^{-v-1} \frac{\partial}{\partial \rho} \left[\rho^{v+1} \left\{\rho^{v} \frac{\partial}{\partial \rho} (\rho^{-v} f) \right\}\right]$$

hal ini disebabkan karena :

$$\rho^{-v-1} \frac{\partial}{\partial \rho} \left\{\rho^{2v+1} \frac{\partial}{\partial \rho} (\rho^{-v} f) \right\}$$

$$= \rho^{-v-1} \frac{\partial}{\partial \rho} \left\{\rho^{2v+1} \left[-v\rho^{-v-1} f(\rho) + \rho^{v}\frac{\partial f}{\partial \rho} \right] \right\}$$

$$= \rho^{-v-1}\left[-v^2\rho^{v-1}f - v\rho^{v} \frac{\partial f}{\partial \rho} + (v+1)\rho^{v+1} \frac{\partial f}{\partial \rho} + \rho^{v} \frac{\partial f}{\partial \rho} + \rho^{v+1} \frac{\partial^2 f}{\partial \rho^2} \right]$$

$$= \rho^{-v-1}\left[-v^2\rho^{v-1}f + \rho^{v} \frac{\partial f}{\partial \rho} + \rho^{v+1} \frac{\partial^2 f}{\partial \rho^2} \right]$$

$$Bvf = \frac{\partial^2 f}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial f}{\partial \rho} - \frac{v^2 f}{\partial \rho^2}$$

Maka dari persamaan (3-4-4)

$$Hv\,[Bv\ f;\ \varphi] = -\varphi^2\ Hv\left[f;\ \varphi\right] \qquad (3\text{-}4\text{-}6)$$

Pandang operator Laplacian tiga dimensi yang didefinisikan dengan :

$$\Delta_3 = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}$$

Substitusi $x = r\cos\phi$; $y = r\sin\phi$ dan $Z = z$

untuk memudahkan penulisan maka :

$$\frac{\partial U}{\partial x} = U_x\ ;\quad \text{dan}\quad \frac{\partial^2 U}{\partial x\,;\,\partial y} = U_{x;y};$$

sehingga :

$$U_x = U_r \cdot r_x + U_\phi \cdot \phi_x$$

$$U_{xx} = U_{rx}\cdot r_x + U_r\cdot r_{xx} + U_{\phi x}\cdot \phi_x + U_\phi\cdot \phi_{xx}$$

$$U_{xr} = U_{rr}\cdot r_x + U_{\phi r}\cdot \phi_x$$

$$U_{x\phi} = U_{\phi r}\cdot r_x + U_{\phi\phi}\cdot \phi_x$$

$$r = (x^2 + y^2)^{1/2} \text{ dan } \phi = \text{arc tan } \frac{y}{x}$$

$$r_x = \frac{x}{r} \; ; \; r_{xx} = \frac{y^2}{r^3} \; ; \; \phi_x = \frac{-y}{r^2} \text{ dan}$$

$$\phi_{xx} = \frac{2xy}{r^4}$$

sehingga :

$$U_{xx} = \frac{x^2}{r^2} U_{rr} - \frac{2xy}{r^3} U_{r\phi} + \frac{y^2}{r^4} U_{\phi\phi} + \frac{y^2}{r^3} U_r + \frac{2xy}{r^3} U_\phi$$

dengan cara yang sama didapat :

$$U_{yy} = \frac{y^2}{r^2} U_{rr} - \frac{2xy}{r^3} U_{r\phi} + \frac{x^2}{r^4} U_{\phi\phi} + \frac{x^2}{r^3} U_r - \frac{2xy}{r^4} U_\phi$$

Kemudian disubstitusikan kedalam operator Laplacian

$$\Delta_3 = U_{xx} + U_{yy} + U_{zz}$$

$$\Delta_3 = U_{rr} + \frac{1}{r} U_r + \frac{1}{r^2} U_{\phi\phi} + U_{zz}$$

sehingga diperoleh operator Laplacian dengan koordinat silinder :

$$\Delta_3 = \frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \emptyset^2} + \frac{\partial^2 U}{\partial z^2}$$

dan jika tak ada ketergantungan ø atau keadaan simetri maka dapat ditulis :

$$\Delta_a = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{v^2}{\partial z^2} \qquad (3\text{-}4\text{-}7)$$

dari definisi jelas bahwa :

$$\Delta_3 U(\rho,z)\, \rho^{iv\phi} = (B_v + D^2)\, U(\rho,z)\, \rho^{iv\phi}$$

$$\Delta_a\, U(r,z) = (B_o + D^2)\, U(\rho,z)$$

dimana D menyatakan operator $\partial/\partial z$ dan dari persamaan (3-4-6) dan (3-4-7) maka

$$Hv\left[\Delta_3 U(\rho,z)\ e^{iv\phi};\ \rho \longrightarrow \varphi\right]$$

$$= Hv\left[(Bv + D^2)\ U(\rho,z)\ e^{iv\phi};\ \rho \longrightarrow \varphi\right]$$

$$= Hv\left[Bv\ U(\rho,z)\ e^{iv\phi};\ \rho \longrightarrow \varphi\right]$$

$$+ Hv\left[D^2\ U(\rho,z)\ e^{iv\phi};\ \rho \longrightarrow \varphi\right]$$

$$= -\varphi^2\ Hv\left[U(\rho,z)\ e^{iv\phi};\ \rho \longrightarrow \varphi\right]$$

$$+ D^2\ Hv\left[U(\rho,z)\ e^{iv\phi};\ \rho \longrightarrow \varphi\right]$$

$$= (D^2 - \varphi^2)\ Hv\left[U(\rho,z)\ e^{iv\phi};\ \rho \longrightarrow \varphi\right]$$

$$= (D^2 - \varphi^2)\ \bar{U}v(\varphi,z)\ e^{iv\phi}$$

dengan $\bar{U}v(\varphi,z)\ e^{iv\phi} = Hv\left[U(\rho,z)\ e^{iv\phi};\ \rho \longrightarrow \varphi\right]$

didapat :

$$Hv\left[\Delta_3\ U(\rho,z)\ e^{iv\phi};\ \rho \longrightarrow \varphi\right] = (D^2 - \varphi^2)\ \bar{U}v(\varphi,z)\ e^{iv\phi} \tag{3-4-8}$$

demikian juga :

$$Ho\left[\Delta_a\ U(\rho,z);\ \rho \longrightarrow \varphi\right] = Ho\left[(Bo + D^2)\ U(\rho,z);\ \rho \longrightarrow \varphi\right]$$

$$= Ho\left[Bo\ U(\rho,z);\ \rho \longrightarrow \varphi\right] + Ho\left[D^2\ U(\rho,z);\ \rho \longrightarrow \varphi\right]$$

$$= -\varphi^2 Ho\left[U(\rho,z);\ \rho \longrightarrow \varphi\right] + D^2\ Ho\left[U(\rho,z);\ \rho \longrightarrow \varphi\right]$$

$$= (D^2 - \varphi^2)\ Ho\left[U(\rho,z);\ \rho \longrightarrow \varphi\right]$$

$$= (D^2 - \varphi^2)\ \bar{U}o(\varphi,z) \tag{3-4-9}$$

dengan $Ho\left[U(\rho,z);\ \rho \rightarrow \varphi \right] = \bar{U}o(\varphi,z)$

Dalam pembicaraan masalah hidrodinamic dan dalam teori elasticity diperlukan rumus-rumus :

$$Hv\left[\Delta_3^2\, U(\rho,z)\ \rho^{iv\emptyset};\ \rho \rightarrow \varphi\right] = (D^2-\varphi^2)^2\ \bar{U}v(\varphi,z)e^{iv\emptyset} \quad (3-4-10)$$

$$Ho\left[\Delta_a^2\, U(\rho,z)\ ;\ \rho \rightarrow \varphi\right] = (D^2-\varphi^2)^2\ \bar{U}o(\varphi,z) \quad (3-4-11)$$

dengan pembuktian :

$$Hv\left[\ \Delta_3^2\ U(\rho,z)\ \rho^{iv\emptyset};\ \rho \rightarrow \varphi\right]$$

$$= Hv\left[\ \Delta_3\left\{\ \Delta_3 U(\rho,z)\ \rho^{iv\emptyset}\right\};\ \rho \rightarrow \varphi\right]$$

$$= Hv\left[\ \Delta_3\ (B_v^2 + D^2)U(\rho,z)\ \rho^{iv\emptyset};\ \rho \rightarrow \varphi\right]$$

$$= (-\varphi^2 + D^2)^2\ Hv\left[\ U(\rho,z)\ \rho^{iv\emptyset};\ \rho \rightarrow \varphi\right]$$

$$= (-\varphi^2 + D^2)^2\ \bar{U}(\rho,z)\ \rho^{iv\emptyset}$$

demikian juga :

$$Ho\left[\ \Delta_a^2\ U(\rho,z);\ \rho \rightarrow \varphi\right] = Ho\left[\ \Delta a\left\{\ \Delta a U(\rho,z)\ \right\};\ \rho \rightarrow \varphi\right]$$

$$= Ho\left[\ \Delta a\ (B_o - \varphi^2)\ U(\rho,z);\ \rho \rightarrow \varphi\ \right]$$

$$= (D^2 - \varphi^2)^2\ Ho\left[\ U(\rho,z);\ \rho \rightarrow \varphi\ \right]$$

$$= (D^2 - \varphi^2)^2\ \bar{U}o(\rho,z)$$

3.5. Transformasi Hankel dari fungsi elementer

Dalam sub bab ini akan diperlihatkan perhitungan transformasi Hankel dari beberapa fungsi elementer.

Fungsi Bessel yang terdapat dalam transformasi Hankel dengan asumsi : $\mu > v \geq 0$, dan ($x > 0$, $a > 0$) dengan demikian :

$$Hv\left[x^{v}\ (a^2 - x^2)^{\mu-v-1}\ H(a-x);\ \varphi\right]$$

$$= \int_0^a x^{v+1}(a^2 - x^2)^{\mu-v-1} Jv(\varphi x)dx$$

$$= \int_0^a x^{v+1}(a^2 - x^2)^{\mu-v-1} \sum_{r=0}^{\infty} \frac{(-1)^r\ (x\varphi/2)^{2r+v}}{r!\ \Gamma(r+v+1)}\, dx$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r\ (\varphi/2)^{v+2r}}{r!\ \Gamma(r+v+1)} \int_0^a x^{2v+2r+1}(a^2 - x^2)^{\mu-v-1}dx$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r\ (\varphi/2)^{v+2r}}{r!\ \Gamma(r+v+1)}\ a^{3a-2v-2} \int_0^a x^{2v+2r}(1 - x^2/a^2)^{\mu-v-1} x\,dx$$

Substitusi :

$x^2 = a^2y$ maka $2xdx = a^2dy$

$$xdx = \frac{a^2}{2}\ dy$$

$x = 0$ maka $y = 0$

$x = a$ maka $y = 1$

sehingga :

$$Hv\left[x^{v}\ (a^2 - x^2)^{\mu-v-1}\ H(a-x);\ \varphi\right]$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r\ (\varphi/2)^{v+2r}}{r!\ \Gamma(r+v+1)}\ \frac{a^{2\mu+2r}}{2} \int_0^1 y^{v+r}(1 - y)^{\mu-v-1}dy$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r\ (\varphi/2)^{v+2r}}{r!\ \Gamma(r+v+1)}\ \frac{a^{2\mu+2r}}{2}\ B(v+r+1),\mu-v)$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r\ (\varphi/2)^{v+2r}}{r!\ \Gamma(r+v+1)}\ \frac{a^{2\mu+2r}}{2}\ \frac{\Gamma(v+r+1)\ \Gamma(\mu-v)}{\Gamma(\mu+r+1)}$$

$$= 2^{\mu-v-1}\Gamma(\mu-v)\ a^{\mu}\ \varphi^{v-\mu} \sum_{r=0}^{\infty} \frac{(-1)^r (a\varphi/2)^{v+2r}}{r!\ \Gamma(r+v+1)}$$

$$= 2^{\mu-v-1}\Gamma(\mu-v)\ a^{\mu}\ \varphi^{v-\mu}\ J\mu(\varphi a)$$

dengan demikian transformasi Hankel menjadi :

$$Hv\left[x^{v}\ (a^2 - X^2)^{\mu-v-1}\ H(a-x);\ \varphi\right]$$

$$= 2^{\mu-v-1}\Gamma(\mu-v)\ a^{\mu}\ \varphi^{v-\mu}\ J\mu(\varphi a) \qquad (3\text{-}5\text{-}1)$$

dengan $\mu > v \geq 0$

Dengan menggunakan invers transformasi Hankel akan didapat jika $\mu > v \geq 0$

$$2^{\mu-v-1}\Gamma(\mu-v)\ a^{\mu}\ Hv\left[\varphi^{v-\mu}\ J\mu(\varphi a);\ x\right]$$

$$= x^{v}\ (a^2 - X^2)^{\mu-v-1}\ H(a-x)$$

$$Hv\left[\varphi^{v-\mu}\ J\mu(\varphi a);\ x\right] = \frac{x^{v}\ (a^2 - X^2)^{\mu-v-1}\ H(a-x)}{2^{\mu-v-1}\Gamma(\mu-v)\ a^{\mu}} \qquad (3\text{-}5\text{-}2)$$

dengan $a > 0$, $\mu > v \geq 0$

dari definisi :

$$Hv\left[\varphi^{v-\mu}\ J\mu(\varphi a);\ x\right] = \int_0^{\infty} \varphi^{v-\mu+1} J\mu(\varphi a) Jv(\varphi x)d\varphi$$

maka :

$$\int_0^{\infty} \varphi^{v-\mu+1} J\mu(\varphi a) Jv(\varphi x)d\varphi = \frac{x^{v}\ (a^2 - X^2)^{\mu-v-1}\ H(a-x)}{2^{\mu-v-1}\Gamma(\mu-v)\ a^{\mu}}$$

ambil $\mu = v + 1$ dalam persamaan (3-5-1) dan (3-5-2) berturut-turut didapat :

$$Hv\left[x^{v}H(a-x);\ \varphi\right] = \frac{a^{v+1}}{\varphi} Jv{+}1(\varphi a), \quad a > 0 \qquad (3\text{-}5\text{-}4)$$

$$H_v\left[\varphi^{-1}J_{v+1}(a\varphi);\ x\right] = \frac{x^v}{a^{v+1}}H(a-x), \quad a > 0 \qquad (3\text{-}5\text{-}5)$$

dengan jalan yang sama jika diambil $\mu = v + \frac{1}{2}$ pada persamaan yang sama maka

$$H_v\left[\frac{x^v H(a-x)}{(a^2-x^2)^{1/2}};\ \varphi\right] = \sqrt{\frac{\pi}{2\varphi}}\ a^{v+1/2}J_{v+1/2}(\varphi a) \quad (3\text{-}5\text{-}6)$$

$$H_v\left[\varphi^{-1/2}J_{v+1/2}(a\varphi);x\right] = \sqrt{\frac{\pi}{2}}\ \frac{x^v\ H(a-x)}{a^{v+1/2}\ (a^2-x^2)^{1/2}} \quad (3\text{-}5\text{-}7)$$

Jika diambil $v = 0$ dalam persamaan (3-5-5) dan (3-5-7) didapat :

$$H_o\left[\varphi^{-1}J_1(a\varphi);\ x\right] = \frac{H(a-x)}{a}, \quad a > 0 \qquad (3\text{-}5\text{-}8)$$

$$H_o\left[\varphi^{-1/2}J_{1/2}(a\varphi);\ x\right] = \sqrt{\frac{2}{\pi}}\ \frac{H(a-x)}{a^{1/2}(a^2-\ x^2)^{1/2}} \quad (3\text{-}5\text{-}9)$$

Karena $J_{1/2}(a\varphi) = \left(\frac{2}{\pi a\varphi}\right)^{1/2}$ sin (ax) maka persamaan (3-5-9) menjadi :

$$H_o\left[\varphi^{-1}\sin\ (a\varphi);\ x\right] = \frac{H(a-x)}{(a^2-\ x^2)^{1/2}}, \quad a > 0 \qquad (3\text{-}5\text{-}9)'$$

Jika diambil $a = t$ maka :

$$H_o\left[\varphi^{-1}\sin\ (t\varphi);\ x\right] = \frac{H(t-x)}{(t^2-\ x^2)^{1/2}}$$

$$\int_0^{\infty} \sin\ (t\varphi)J_o(\varphi x)d\varphi = \frac{H(t-x)}{(t^2-\ x^2)^{1/2}}$$

Pandang :

$$\mathcal{F}_s\left[J_o(\varphi x);\ t\right] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} J_o(\varphi x)\ \sin t\varphi\ \ d\varphi$$

Sehingga

$$\mathcal{F}_s\left[J_o(\varphi x);\ t\right] = \sqrt{\frac{2}{\pi}}\ \frac{H(t-x)}{(t^2-\ x^2)^{1/2}}$$

Karena $\mathcal{F}_s = \mathcal{F}_s^{-1}$ maka

$$Jo(\varphi x) = \sqrt{\frac{2}{\pi}}\, \mathcal{F}_s \left[\frac{H(t-x)}{(t^2 - x^2)^{1/2}} ; \varphi \right]$$

$$= \frac{2}{\pi} \int_x^{\infty} \frac{\sin(\varphi t)}{(t^2 - x^2)^{1/2}}\, dt$$

$$\int_x^{\infty} \frac{\sin(\varphi t)}{(t^2 - x^2)^{1/2}}\, dt = \frac{\pi}{2}\, Jo(\varphi x) \qquad (3-5-10)$$

Selanjutnya dibicarakan transformasi Hankel dari fungsi eksponensial yang dengan menggunakan definisi transformasi Hankel maka :

$$Hv \left[e^{-px} f(x) ; \varphi \right] = \int_0^{\infty} \left[x\, f(x) Jv(\varphi x) \right] e^{-px}\, dx$$

Jika $f(x) = x^{v-1}$. maka

$$Hv \left[e^{-px} x^{v-1} ; \varphi \right] = \int_0^{\infty} x^v\, e^{-px} Jv(\varphi x)\, dx$$

$$= \int_0^{\infty} x^v e^{-px} \sum_{r=0}^{\infty} \frac{(-1)^r\, (x\varphi/2)^{2r+v}}{r!\, \Gamma(r+v+1)}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r\, 2^{-2r-v} \varphi^{2r+v}}{r!\, \Gamma(r+v+1)} \int_0^{\infty} x^{2r+2v}\, \rho^{-px}\, dx$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r\, 2^{-2r-v} \varphi^{2r+v}}{r!\, \Gamma(r+v+1)}\, p^{-2r-2v-1} \int_0^{\infty} x^{2r+2v}\, e^{-x}\, dx$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r\, 2^{-v-2v} \varphi^{2r+v}\, p^{-2r-2v-1}}{r!\, \Gamma(r+v+1)}\, \Gamma(2r+2v+1)$$

Karena $\Gamma(2r+2v+1) = 2^{2v+2r} \dfrac{\Gamma(r+v+1/2)\, \Gamma(r+v+1)}{\Gamma(1/2)}$

sehingga :

$$Hv\left[x^{v-1}e^{-px};\varphi\right]$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r\ 2^{-v-2v}\varphi^{2r+v}\ p^{-2v-2r-1}}{r!\ \Gamma(r+v+1)}\ \Gamma(2r+2v+1)$$

Karena $(1-x)^{-\alpha} = \sum_{r=0}^{\infty} \Gamma\ \frac{(\alpha+r)\ x^r}{r!\ \Gamma(\alpha)}$

$$\Gamma(\alpha)(1-x)^{-\alpha} = \sum_{r=0}^{\infty}\Gamma\ \frac{(\alpha+r)\ x^r}{r!} \qquad (3-5-11)$$

ambil $\alpha = v + \frac{1}{2}$ dan $x = -\left(\frac{p}{\varphi}\right)^{-2}$

$$\Gamma(v+\tfrac{1}{2})\left[1+\left[\frac{p}{\varphi}\right]^{-2}\right]^{-v-1/2} = \sum_{r=0}^{\infty}\Gamma\ \frac{(v+1/2+r)\left[-(p/\varphi)^{-2}\right]^r}{r!}$$

maka

$$Hv\left[x^{v-1}e^{-px};\varphi\right] = \frac{2^v\varphi^v p^{-2v-1}}{\Gamma(1/2)}\ \Gamma(v+\tfrac{1}{2})\left[1+\frac{\varphi^2}{p^2}\right]^{-v-1/2}$$

$$= \frac{2^v\varphi^v\Gamma(v+1/2)}{\Gamma(1/2)}\ \frac{p^{-2v-1}p^{2v+1}}{(p^2+\varphi^2)^{v+1/2}}$$

$$= \frac{2^v\varphi^v\Gamma(v+1/2)}{\Gamma(1/2)(p^2+\varphi^2)^{v+1/2}}$$

Dengan demikian :

$$Hv\left[x^{v-1}e^{-px};\varphi\right] = \frac{2^v\varphi^v\Gamma(v+1/2)}{\Gamma(1/2)(p^2+\varphi^2)^{v+1/2}}\ , v > -\frac{1}{2} \quad (3-5-12)$$

Demikian juga untuk $f(x) = x^v \qquad (x>0,\ p>0)$

$$Hv\left[x^v\ e^{-px};\varphi\right] = \int_0^{\infty} x^{v+1}Jv(\varphi x)\ e^{-px}\ dx$$

$$= \int_0^{\infty} x^{v+1}\ e^{-px}\ \sum_{r=0}^{\infty}\frac{(-1)^r\ (\varphi x/2)^{2r+v}}{r!\ \Gamma(r+v+1)}$$

$$= \sum_{r=0}^{\infty}\frac{(-1)^r\ 2^{-2r-v}\varphi^{2r+v}}{r!\ \Gamma(r+v+1)}\int_0^{\infty} x^{2r+2v+1}\ e^{-px}\ dx$$

$$= \sum_{r=0}^{\infty}\frac{(-1)^r\ 2^{-2r-v}\ \varphi^{2r+v}\ p^{-2v-2r-2}}{r!\ \Gamma(r+v+1)}\int_0^{\infty} x^{2r+2v+1}\ e^{-x}\ dx$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r \; 2^{-2r-v} \; \varphi^{2r+v} \; p^{-2v-2r-2}}{r! \; \Gamma(r+v+1)} \Gamma(2v+2r+2)$$

Karena :

$$\frac{2^{-v-2r} \; \Gamma(2v+2r+2)}{\Gamma(v+r+1)} = \frac{2^{v+1} \; \Gamma(v+r+3/2)}{\Gamma(1/2)}$$

dan karena persamaan (3-5-11) maka :

Jika $\alpha = v + 3/2$ dan $x = -(p/\varphi)^{-2}$ didapat :

$$\Gamma(v+3/2)\left[1+ \frac{\varphi^2}{p^2}\right]^{-v-3/2} = \sum_{r=0}^{\infty} \frac{\Gamma(v+r+3/2)}{r!} \left[-(p/\varphi)^{-2}\right]^r$$

$$Hv \; \left[x^v \; e^{-px};\varphi\right]$$

$$= \frac{2^{v+1}\varphi^{v}\pi^{-1/2}p^{-2v-2}\Gamma(v+3/2)(p^2+\varphi^2)^{-v-3/2}}{p^{-2v-3}}$$

$$= \frac{2^{v+1} \; \varphi^v \; p \; \Gamma(v+3/2)}{\Gamma(1/2)(p^2+\varphi^2)^{v+3/2}}$$

sehingga didapat rumus :

$$Hv \; \left[x^v \; e^{-px};\varphi\right] = \frac{2^{v+1} \; \varphi^v \; p \; \Gamma(v+3/2)}{\Gamma(1/2)(p^2+\varphi^2)^{v+3/2}} \qquad (3\text{-}5\text{-}13)$$

$(x>0, \; p>0)$

Sekarang pandang transformasi Hankel :

$$Hv \; \left[f(x)^v \; e^{-1/4px^2};\varphi\right] = \int_0^{\infty} x \; f(x) \; e^{-px^2/4} Jv(xu) \; dx$$

Jika diambil $f(x) = x^v$ maka

$$Hv \; \left[x^v \; e^{-px^2/4};\varphi\right] = \int_0^{\infty} x^{v+1} \; e^{-px^2/4} \; Jv(xu) \; dx$$

$$= \int_0^{\infty} x^{v+1} \; e^{-px^2/4} \sum_{r=0}^{\infty} \frac{(-1)^r \; \varphi^{2r+v} \; x^{2r+v}}{r! \; 2^{2r+v} \; \Gamma(r+v+1)} \, dx$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r \; \varphi^{2r+v}}{r! \; 2^{2r+v} \; \Gamma(r+v+1)} \int_0^{\infty} x^{2r+2v+1} \; e^{-px^2/4} dx$$

substitusi $x^2/4 = u \;\longrightarrow\; x = 2u^{1/2}$; $x \; dx = 2 \; du$

Sehingga :

$$Hv\left[x^{v}\, e^{-1/4px^2};\varphi\right]$$

$$= 2^{v+1}\,\varphi^{v}\sum_{r=0}^{\infty}\frac{(-1)^r\,\varphi^{2r}}{r!\,\Gamma(r+v+1)}\,p^{-v-r-1}\int_{0}^{\infty}s^{r+v}\,e^{-s}\,ds$$

$$= \frac{2^{v+1}\,\varphi^{v}}{p^{v+1}}\sum_{r=0}^{\infty}\frac{(-1)^r\,\Gamma(r+v+1)}{r!\,\Gamma(r+v+1)}\,p^{-r}\,\varphi^{2r}$$

$$= \frac{2^{v+1}\,\varphi^{v}}{p^{v+1}}\sum_{r=0}^{\infty}\frac{(-1)^r(\varphi^2/p)^r}{r!}$$

$$= \frac{2^{v+1}\,\varphi^{v}}{p^{v+1}}\,e^{-\varphi^2/p} \qquad (3-5-14)$$

Jika $p = 4/a^2$ maka

$$Hv\left[x^{v}\,e^{-x^2/a^2};\ \varphi\right] = \left(\tfrac{1}{2}\,a^2\right)^{v+1}\varphi^{v}\,e^{-1/4\varphi^2 a^2} \qquad (3-5-15)$$

dan jika $a^2 = 2$ maka

$$Hv\left[x^{v}\,e^{-x^2/2};\ \varphi\right] = \varphi^{v}\,e^{-1/2\varphi^2} \qquad (3-5-16)$$

Hasil ini memperlihatkan bahwa $x^{v}\,e^{-1/2x^2}$ adalah self reciprocal dari transformasi Hankel order v.

3.6. Transformasi Abel

Didefinisikan transformasi Abel A_1 dan A_2 dalam bentuk :

$$\hat{f}_1(x) = A_1\left[f(t);x\right] = \sqrt{\frac{2}{\pi}}\int_{0}^{x}\frac{f(t)\,dt}{(x^2-t^2)^{1/2}}\ ,\ x > 0 \qquad (3-6-1)$$

$$\hat{f}_2(x) = A_2\left[f(t);x\right] = \sqrt{\frac{2}{\pi}}\int_{x}^{\infty}\frac{f(t)\,dt}{(t^2-x^2)^{1/2}}\ ,\ x > 0 \qquad (3-6-2)$$

Pandang persamaan (3-6-1) dan (3-6-2) sebagai persamaan integral $f(t)$. Berdasarkan persamaan (2-10-14) maka persamaan (3-6-1) mempunyai penyelesaian :

$$f(t) = \sqrt{\frac{2}{\pi}} \frac{d}{dt} \int_0^t \frac{x \hat{f}_1(x) \, dx}{(t^2 - x^2)^{1/2}} \quad , \ t > 0$$

yang mana dapat dinyatakan dengan :

$$f(t) = A_1^{-1}[\hat{f}_1(x); t] = Dt \, A_1[x \hat{f}_1(x); t] \qquad (3\text{-}6\text{-}3)$$

dimana Dt menyatakan operator differensial $\frac{d}{dt}$.

Demikian juga persamaan (3-6-2) mempunyai penyelesaian berdasarkan persamaan (2-10-16) yaitu :

$$f(t) = -\sqrt{\frac{2}{\pi}} \frac{d}{dt} \int_t^{\infty} \frac{x \hat{f}_2(x) \, dx}{(x^2 - t^2)^{1/2}} \quad , \ t > 0$$

$$f(t) = A_2^{-1}[\hat{f}_2(x); t] = -Dt \, A_2[x \hat{f}_2(x); t] \qquad (3\text{-}6\text{-}4)$$

Banyak integral yang dapat dinyatakan dengan transformasi Abel, sebagai contoh, dapat menginterpretasikan persamaan (2-8-5) dan (3-5-10) dalam bentuk :

$$A_1[\cos(\varphi t); x] = \sqrt{\frac{\pi}{2}} \, J_0(\varphi x) \qquad (3\text{-}6\text{-}5)$$

$$A_2[\sin(\varphi t); x] = \sqrt{\frac{\pi}{2}} \, J_0(\varphi x) \qquad (3\text{-}6\text{-}6)$$

Jika persamaan (3-6-5) didifferensialkan terhadap φ didapat rumus :

$$A_1[t \sin(\varphi t); x] = \sqrt{\frac{\pi}{2}} \, x \, J_1(\varphi x) \qquad (3\text{-}6\text{-}7)$$

dan dengan menggunakan hasil ini dalam persamaan (3-6-3) didapat :

$$A_1^{-1}[\sin(\varphi x); t] = \sqrt{\frac{\pi}{2}} \, \varphi t \, J_0(\varphi t) \qquad (3\text{-}6\text{-}8)$$

dengan jalan yang sama dari persamaan (3-6-4) dan (3-6-6) diperoleh :

$$A_2^{-1}\left[\frac{\sin(\varphi x)}{x}\,;\,t\right] = -Dt\; A_2\left[\sin(\varphi x)\,;\,t\right]$$

$$= -\sqrt{\frac{\pi}{2}}\; Dt\; J_0(\varphi t)$$

$$= \sqrt{\frac{\pi}{2}}\;\varphi\; J_1(\varphi t) \qquad (3\text{-}6\text{-}9)$$

3.7. Hubungan Antara Transformasi Fourier dan Hankel

Dari persamaan (3-6-5) dan (3-6-6) dapat pula dinyatakan dengan :

$$\int_0^x \frac{\cos(\varphi t)}{(x^2-t^2)^{1/2}}\,dt = \frac{\pi}{2}\,J_0(\varphi x) \quad ,(x>0,\ \varphi>0) \qquad (3\text{-}7\text{-}1)$$

$$\int_x^\infty \frac{\sin(\varphi t)}{(t^2-x^2)^{1/2}}\,dt = \frac{\pi}{2}\,J_0(\varphi x) \quad ,(x>0,\ \varphi>0) \qquad (3\text{-}7\text{-}2)$$

dengan menggunakan transformasi Abel dapat dibuktikan bahwa $H_v = H_v^{-1}$ untuk $v = 0$ dan $v = 1$, yang sering digunakan dalam aplikasinya, untuk $H_0 = H_0^{-1}$

Bukti :

Pandang suatu transformasi

$$\mathcal{F}_s\Big[A_1\big[t\,f(t);\,x\big];\,\varphi\Big] = \sqrt{\frac{2}{\pi}}\int_0^\infty A_1\big[t\,f(t);\,x\big]\,\sin(\varphi x)\,dx$$

$$= \frac{2}{\pi}\int_0^\infty\int_0^x \frac{t\,f(t)\,\sin(\varphi x)}{(x^2-t^2)^{1/2}}\,dt\,dx$$

dari persamaan (2-11-2) dapat ditulis :

$$\mathcal{F}_s\Big[A_1\big[t\,f(t);\,x\big];\,\varphi\Big] = \frac{2}{\pi}\int_0^\infty t\,f(t)\int_t^\infty \frac{\sin(\varphi x)}{(x^2-t^2)^{1/2}}\,dx\,dt$$

$$= \int_0^\infty t\,f(t)\,J_0(\varphi x)\,dt$$

$$= \bar{f}_0(\varphi)$$

$$\mathcal{F}_s\left[A_1[t\ f(t);\ x];\ \varphi\right] = \bar{f}_0(\varphi) = H_0\ [f(t);\ \varphi]$$

Karena $\mathcal{F}_s = \mathcal{F}_s^{-1}$ maka

$$A_1[t\ f(t);\ x] = \mathcal{F}_s\ [\ \bar{f}_0(\varphi);\ x]$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} \bar{f}_0(\varphi)\ \sin\ (\varphi x)\ d\varphi$$

dengan menggunakan invers transformasi Abel maka didapat :

$$t\ f(t) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \bar{f}_0(\varphi)\ A_1^{-1}\ [\sin\ (\varphi x);\ t]\ d\varphi$$

dari persamaan (3-6-8) maka

$$t\ f(t) = \int_0^{\infty} \bar{f}_0(\varphi)\ \varphi t\ J_0(\varphi t)\ d\varphi$$

$$f(t) = \int_0^{\infty} \varphi\ \bar{f}_0(\varphi)\ J_0(\varphi t)\ d\varphi$$

$$f(t) = H_0\ [\ \bar{f}_0(t);\ \varphi]$$

sehingga terbukti $H_0 = H_0^{-1}$

Sekarang kita buktikan $H_1 = H_1^{-1}$

Bukti :

Pandang suatu transformasi :

$$\mathcal{F}_s\ [x\ \hat{f}_2(x);\ \varphi] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} x\ \hat{f}_2(x)\ \sin\ (\varphi x)\ dx$$

$$= \frac{2}{\pi} \int_0^{\infty} x\ \sin\ (\varphi x) \int_x^{\infty} \frac{f(t)\ dt}{(t^2 - x^2)^{1/2}}\ dx.$$

$$= \frac{2}{\pi} \int_0^{\infty} f(t) \int_0^{t} \frac{x\ \sin\ (\varphi x)}{(t^2 - x^2)^{1/2}}\ dx\ dt$$

Karena $A_1[x\ \sin\ (\varphi x);\ t] = \sqrt{\frac{\pi}{2}}\ t\ J_1(\varphi t)$ berarti

$$\sqrt{\frac{2}{\pi}} \int_0^t \frac{x \sin(\varphi x)}{(t^2 - x^2)^{1/2}}\, dx = \sqrt{\frac{\pi}{2}}\ t\ J_1(\varphi t)$$

$$\int_0^t \frac{x \sin(\varphi x)}{(t^2 - x^2)^{1/2}}\, dx = \frac{\pi}{2}\ t\ J_1(\varphi t)$$

maka

$$\mathcal{F}_s \left[x\ \hat{f}_2(x);\ \varphi\right] = \int_0^\infty t\ f(t)\ J_1(\varphi x)\ dt$$

$$= \bar{f}_1(\varphi) = H_1\left[\ f(t);\ \varphi\right]$$

Karena $\mathcal{F}_s = \mathcal{F}_s^{-1}$ maka

$$x\ \hat{f}_2(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \bar{f}_1(\varphi) \sin(\varphi x)\ d\varphi$$

$$\hat{f}_2(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \bar{f}_1(\varphi)\ \frac{\sin(\varphi x)}{x}\ d\varphi$$

$$A_2\left[\ f(t);\ x\right] = \sqrt{\frac{2}{\pi}} \int_0^\infty \bar{f}_1(\varphi)\ \frac{\sin(\varphi x)}{x}\ d\varphi$$

dengan menggunakan invers transformasi Abel maka

$$f(t) = \sqrt{\frac{2}{\pi}} \int_0^\infty \bar{f}_1(\varphi)\ A_2^{-1} \left[\frac{\sin(\varphi x)}{x}\ ;\ t\right] d\varphi$$

$$f(t) = \int_0^\infty \varphi\ \bar{f}_1(\varphi)\ J_1(\varphi t)\ d\varphi$$

$$f(t) = H_1\left[\ \bar{f}_1(\varphi);\ t\right]$$

sehingga terbukti $H_1 = H_1^{-1}$

Untuk mendapatkan hubungan antara transformasi Fourier dan Hankel, terlebih dulu dari persamaan (3-7-1) dan (3-7-2) didapat :

$$\mathcal{F}_c\left[\ J_0(\varphi\rho);\ \varphi \longrightarrow x\right] = \sqrt{\frac{2}{\pi}}\ \frac{H(\rho - x)}{(\rho^2 - x^2)^{1/2}} \qquad (3\text{-}7\text{-}3)$$

$$\mathcal{F}_s\left[\, Jo(\varphi\rho);\ \varphi \rightarrow x\right] = \sqrt{\frac{2}{\pi}}\ \frac{H(x-\rho)}{(x^2-\rho^2)^{1/2}} \qquad (3\text{-}7\text{-}4)$$

pandang transformasi :

$$\mathcal{F}_c\left[\, \bar{f}o(\varphi);\ x\right] = \int_0^{\infty} \rho\ f(\rho)\mathcal{F}_c\left[\, Jo(\varphi\rho);\ \varphi \rightarrow x\right] d\rho$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} \rho\ f(\rho) \frac{H(\rho-x)}{(\rho^2-x^2)^{1/2}}\, d\rho$$

$$= \sqrt{\frac{2}{\pi}} \int_x^{\infty} \frac{\rho\ f(\rho)}{(\rho^2-x^2)^{1/2}}\, d\rho$$

$$\mathcal{F}_c\left[\, \bar{f}o(\varphi);\ x\right] = A_2\left[\, \rho\ f(\rho);\ x\right] \qquad (3\text{-}7\text{-}5)$$

Demikian juga secara sama dapat dibuktikan

$$\mathcal{F}_s\left[\, \bar{f}o(\varphi);\ x\right] = A_1\left[\, \rho\ f(\rho);\ x\right] \qquad (3\text{-}7\text{-}6)$$

Jika $Fs(\varphi) = \mathcal{F}_s\left[\, f(t);\ \varphi\right]$ maka

$$Ho\left[\, \varphi^{-1}\ Fs(\varphi);\ \rho\right] = \int_0^{\infty} Jo(\varphi\rho)\ Fs(\varphi)\ d\varphi$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} Jo(\varphi\rho) \int_0^{\infty} f(t)\ \sin\ (\varphi t)\ dt\ d\rho$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \int_0^{\infty} Jo(\varphi\rho)\ \sin\ (\varphi t)\ d\varphi\ dt$$

dari persamaan (3-7-4) maka

$$Ho\left[\, \varphi^{-1}\ Fs(\varphi);\ \rho\right] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \frac{H(t-\rho)}{(t^2-\rho^2)^{1/2}}\, dt$$

$$= \sqrt{\frac{2}{\pi}} \int_\rho^{\infty} \frac{f(t)}{(t^2-\rho^2)^{1/2}}\, dt$$

sehingga didapat hubungan sebagai berikut :

$$Ho\left[\, \varphi^{-1}\ Fs(\varphi);\ \rho\right] = A_2\left[\, f(t);\ \rho\right] \qquad (3\text{-}7\text{-}7)$$

demikian juga untuk :

$$\mathrm{Ho}\left[\varphi^{-1}\,Fc(\varphi);\ \rho\right] = A_1\left[f(t);\ \rho\right] \qquad (3\text{-}7\text{-}8)$$

3.8. Rumus-Rumus Beltrami

Yang dimaksud dari rumus Beltrami adalah jika :

$$Fs(\varphi) = \bar{f}o(\varphi) \qquad (3\text{-}8\text{-}1)$$

dimana

$$\bar{f}o(\varphi) = \mathrm{Ho}\left[f(x);\ \varphi\right]$$

$$Fs(\varphi) = \mathcal{F}_s\left[f(t);\ \varphi\right]$$

sehingga

$$\mathcal{F}_s\left[f(t);\ \varphi\right] = \mathrm{Ho}\left[f(x);\ \varphi\right]$$

Karena $\mathcal{F}_s = \mathcal{F}_s^{-1}$ maka didapat

$$f(t) = \mathcal{F}_s\left[\mathrm{Ho}\left[f(x);\ \varphi\right];\ t\right]$$

dari persamaan (3-7-6)

$$\mathcal{F}_s\left[\mathrm{Ho}\left[f(x);\ \varphi\right];\ t\right] = A_1\left[x\ f(x);\ t\right]$$

sehingga didapat :

$$f(t) = A_1\left[x\ f(x);\ t\right] \qquad (3\text{-}8\text{-}2)$$

Substitusi persamaan (3-8-1) dan (3-8-2) kedalam persamaan (3-7-7), yaitu :

$$\mathrm{Ho}\left[\varphi^{-1}\,Fs(\varphi);\ \rho\right] = A_2\left[f(t);\ \rho\right]$$

$$\mathrm{Ho}\left[\varphi^{-1}\,\bar{f}o(\varphi);\ \rho\right] = A_2\left[A_1\left[x\ f(x);\ t\right];\ \rho\right] \qquad (3\text{-}8\text{-}3)$$

$$= \sqrt{\frac{2}{\pi}}\int_{\rho}^{\infty}\frac{A_1\left[x\ f(x);\ t\right]}{(t^2-\rho^2)^{1/2}}\,dt$$

$$= \frac{2}{\pi}\int_{\rho}^{\infty}\frac{1}{(t^2-\rho^2)^{1/2}}\int_{0}^{t}\frac{x\ f(x)\,dx}{(t^2-x^2)^{1/2}}\,dt$$

sehingga :

$$\mathrm{Ho}\left[\,\varphi^{-1}\,\bar{f}_o(\varphi);\ \rho\right]$$

$$= \frac{2}{\pi}\int_{\rho}^{\infty}\frac{1}{(t^2-\rho^2)^{1/2}}\int_{0}^{t}\frac{x\,f(x)\,dx}{(t^2-x^2)^{1/2}}\,dt \qquad (3\text{-}8\text{-}4)$$

demikian juga :

$$\mathrm{Ho}\left[\,\varphi\ \ \bar{f}_o(\varphi);\ \rho\right]$$

$$= \frac{-2}{\pi}\frac{d}{d\rho}\int_{\rho}^{\infty}\frac{1}{(t^2-\rho^2)^{1/2}}\frac{d}{dt}\int_{0}^{t}\frac{x\,f(x)\,dx}{(t^2-x^2)^{1/2}}\,dt \qquad (3\text{-}8\text{-}5)$$

Bukti :

dari persamaan (3-5-9) yaitu :

$$\mathrm{Ho}\left[\,\varphi^{-1}\sin(\varphi\rho);\ x\right] = \frac{H(\rho-x)}{(\rho^2-x^2)^{1/2}}$$

dengan menggunakan invers transformasi Hankel maka didapat :

$$\varphi^{-1}\sin(\varphi\rho) = \mathrm{Ho}\left[\frac{H(\rho-x)}{(\rho^2-x^2)^{1/2}}\ ;\ \varphi\right]$$

$$= \int_{0}^{\infty}\frac{x\,H(\rho-x)}{(\rho^2-x^2)^{1/2}}\,\mathrm{Jo}(\varphi x)\,dx$$

$$= \int_{0}^{\rho}\frac{x\,\mathrm{Jo}(\varphi x)}{(\rho^2-x^2)^{1/2}}\,dx$$

$$\mathrm{A}_1\left[\,x\,\mathrm{Jo}(\varphi x);\ \rho\right] = \sqrt{\frac{2}{\pi}}\ \varphi^{-1}\sin(\varphi\rho) \qquad (3\text{-}8\text{-}6)$$

demikian juga karena :

$$\int_{0}^{x}\frac{\cos(\varphi\rho)}{(x^2-\rho^2)^{1/2}}\,d\rho = \frac{\pi}{2}\,\mathrm{Jo}(\varphi x)$$

$$\int_0^{\infty} \frac{H(x-\rho)\cos(\varphi\rho)\,d\rho}{(x^2-\rho^2)^{1/2}} = \frac{\pi}{2} Jo(\varphi x)$$

$$\mathcal{F}_c\left[\frac{H(x-\rho);\ \rho \to \varphi}{(x^2-\rho^2)^{1/2}}\right] = \sqrt{\frac{\pi}{2}}\ Jo(\varphi x)$$

Karena $\mathcal{F}_c = \mathcal{F}_c^{-1}$ maka

$$\frac{H(x-\rho)}{(x^2-\rho^2)^{1/2}} = \sqrt{\frac{2}{\pi}}\ \mathcal{F}_c\ [\,Jo(\varphi x);\ \varphi \to \rho]$$

$$\mathcal{F}_c\ [\,Jo(\varphi x);\ \varphi \to \rho] = \sqrt{\frac{2}{\pi}}\ \frac{H(x-\rho)}{(x^2-\rho^2)^{1/2}}$$

$$\int_0^{\infty} Jo\ (\varphi x)\ \cos\ (\varphi\rho)\ d\varphi = \frac{H(x-\rho)}{(x^2-\rho^2)^{1/2}}$$

$$Ho\ [\,\varphi^{-1}\ \cos\ (\varphi\rho);\ x] = \frac{H(x-\rho)}{(x^2-\rho^2)^{1/2}}$$

selanjutnya dengan cara yang sama pada persamaan (3-8-6) didapat :

$$Az\ [\,x\ Jo(\varphi x);\ \rho] = \sqrt{\frac{2}{\pi}}\ \varphi^{-1}\ \cos\ (\varphi\rho) \qquad (3\text{-}8\text{-}7)$$

sebab itu :

$$Az\ [\rho\ Ho[\varphi\ \bar{f}o(\varphi);\ \rho];\ t]$$

$$= Az\left[\int_0^{\infty} \rho\varphi^2\ \ \bar{f}(\varphi)\ Jo(\varphi\rho)\ d\varphi;\ t\right]$$

$$= \int_0^{\infty} \varphi^2\ \ \bar{f}o(\varphi)\ Az\ [Jo(\varphi\rho)\ d\varphi;\ t]\ d\varphi$$

dari pers (3-8-6) maka

$$Az[\rho\ Ho(\varphi\ \bar{f}o(\varphi);\ \rho);\ t] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \varphi\ \bar{f}o(\varphi)\ \cos(\varphi\rho)\ d\rho$$

$$A_1\left[A_2\left\{(\rho\, Ho(\varphi\, \bar{f}o(\varphi);\ \rho);t\right\};\ x\right]$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \varphi\, \bar{f}o(\varphi)\ A_1[\cos(\varphi t);\ x]\ d\varphi$$

dari persamaan (3-6-5) yaitu

$A_1[\cos\ (\varphi t);\ x] = \sqrt{\frac{\pi}{2}}\ Jo(\varphi x)$ maka

$$A_1\left[A_2\left\{(\rho\, Ho(\varphi\, \bar{f}o(\varphi);\ \rho);t\right\};\ x\right] = \int_0^\infty \varphi\ \bar{f}o(\varphi)\ Jo(\varphi x)\ d\varphi$$

$$A_1\left[A_2\left\{(\rho\, Ho(\varphi\, \bar{f}o(\varphi);\ \rho);t\right\};\ x\right] = f(x) \qquad (3\text{-}8\text{-}8)$$

dengan menggunakan invers transformasi Abel maka :

$$A_2\left\{(\rho\, Ho(\varphi\, \bar{f}o(\varphi);\ \rho);t\right\} = A_1^{-1}\ [\ f(x);\ t]$$

$$Ho\,[\varphi\ \bar{f}o(\varphi);\ \rho] = \rho^{-1}\ A_2^{-1}\ [\ A_1^{-1}\ [\ f(x);\ t];\ \rho]$$

yang mana dapat ditulis dalam bentuk :

$Ho\ [\ \varphi^{-1}\ \bar{f}o(\varphi);\ \rho]$

$$= \frac{-2}{\pi\rho}\ \frac{d}{d\rho} \int_\rho^\infty \frac{1}{(t^2-\ \rho^2)^{1/2}}\ \frac{d}{dt} \int_0^t \frac{x\ f(x)\ dx}{(t^2-\ x^2)^{1/2}}\ dt$$