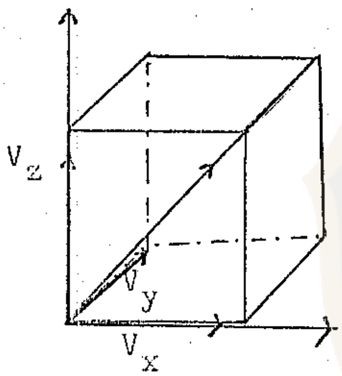
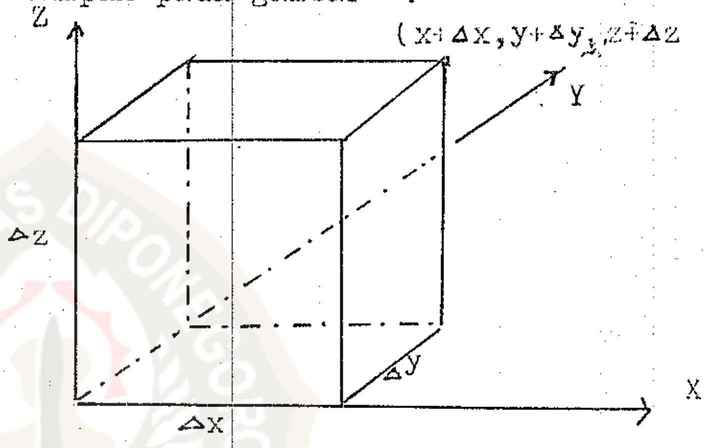
BAB III

PEMBENTUKAN MODEL

3.1 KONTINUITAS ALIRAN FLUIDA

Suatu fluida yang mengalir pada sebuah benda yang berbentuk kubus dengan panjang sisi-sisinya Δx, Δy, Δz dan volume benda yg mempunyai kedudukan tetap dalam ruang. Fluida mengalir dengan kecepatan sebesar $\overline{V}$, seperti tampak pada gambar :





Vektor kecepatan diuraikan sesuai arah koordinatnya yaitu :

V_x = kecepatan pada arah x

V_y = kecepatan pada arah y

V_z = kecepatan pada arah z

Karena keadaan fluida bergerak maka fluida yang masuk pada $x = x$ akan keluar pada $x = x + \Delta x$. berlaku juga bila masuk pada $y = y$ maka akan keluar pada $y = y + \Delta y$ dan juga untuk z, untuk menerapkan persamaan (2) akan terbagi pada arah x, y dan z

Untuk arah x melalui bidang $\Delta z \Delta y$:

Luas bidang : $\Delta z . \Delta y$

Laju alir masa yang masuk : $\Delta z . \Delta y \cdot \rho V_{masuk}$ pd $x = x$

Laju alir masa yang keluar : $\Delta z \Delta y \cdot \rho V_{keluar}$ pd $x = x + \Delta x$

Laju Akumulasi yg tertinggal : $\Delta z \Delta y \Delta x \cdot \frac{\partial \rho}{\partial t}$

Sesuai dengan persamaan (1) yang menyatakan :

Laju alir Akumulasi = Laju alir masuk - Laju alir keluar

maka didapat bentuk :

$$\Delta z \Delta y \Delta x \frac{\partial \rho}{\partial t} = \Delta z \Delta y \cdot \rho V_{\text{masuk pd } x = x} - \Delta z \Delta y \cdot \rho V_{\text{keluar pd } x = x + \Delta x}$$

.... (7)

Untuk arah y melalui bidang $\Delta x \Delta z$

Luas bidang $= \Delta x . \Delta z$

Laju alir masa yang masuk $= \Delta x . \Delta z \cdot \rho V_{\text{masuk pd } y = y}$

Laju alir masa yg keluar $= \Delta x . \Delta z . \rho V_{\text{keluar pd } y = y + \Delta y}$

Laju Akumulasi masa yg tinggal $= \Delta x . \Delta z . \Delta y \frac{\partial \rho}{\partial t}$

Sesuai dengan persamaan (1) yang menyatakan :

Laju alir Akumulasi = Laju alir masuk - Laju alir keluar

maka di-dapat bentuk :

$$\Delta x . \Delta z . \Delta y . \frac{\partial \rho}{\partial t} = \Delta x \Delta z \cdot \rho V_{\text{masuk pd } y=y} - \Delta x \Delta z . \rho V_{\text{keluar pd } y = y + \Delta y}$$

.... (8)

Untuk arah z melalui bidang $\Delta x \Delta y$

Luas bidang $= \Delta x . \Delta y$

Laju alir masa yang masuk $= \Delta x . \Delta y \cdot \rho V_{\text{masuk pd } z=z}$

Laju alir masa keluar $= \Delta x \Delta y \cdot \rho V_{\text{keluar pd } z = z + \Delta z}$

Laju akumulasi masa yang tinggal $= \Delta x \Delta y \Delta z \frac{\partial \rho}{\partial t}$

Sesuai dengan persamaan (1) yang menyatakan :

Laju Akumulasi masa = Laju alir masa yg masuk - Laju alir masa yg keluar.

maka didapat bentuk :

$$\Delta x \Delta y \Delta z \frac{\partial \rho}{\partial t} = \Delta x . \Delta y \cdot \rho V_{\text{masuk pd } z=z} - \Delta x . \Delta y \cdot \rho V_{\text{keluar pd } z = z + \Delta z}$$

.... (9)

Bila dari persamaan (7) , (8) dan (9) dijumlahkan bersama maka didapat :

" Persamaan Akumulasi Masa Seluruhnya "

$$\Delta z.\Delta y \quad \frac{\partial \rho}{\partial t} = \Delta z.\Delta y \cdot \rho V_x|_x - \Delta z.\Delta y \cdot \rho V_x|_{x+\Delta x} \cdot$$

$$\boxed{+}$$

$$\Delta x.\Delta z \quad y \frac{\partial \rho}{\partial t} = \Delta x \Delta z \cdot \rho V_y|_y - \Delta x \Delta z \cdot \rho V_y|_{y+\Delta y} .$$

$$\boxed{+}$$

$$\Delta x.\Delta y \quad z \frac{\partial \rho}{\partial t} = \Delta x \Delta y \cdot \rho V_z|_z - \Delta x \Delta y \cdot \rho V_z|_{z+\Delta z} .$$

dan disederhanakan :

$$\Delta x \cdot \Delta y \cdot \Delta z \frac{\partial \rho}{\partial t} = \Delta y \Delta z \left[\rho V_x|_x - \rho V_x|_{x+\Delta x} \right]$$

$$+ \Delta x \Delta z \left[\rho V_y|_y - \rho V_y|_{y+\Delta y} \right]$$

$$+ \Delta y \Delta x \left[\rho V_z|_z - \rho V_z|_{z+\Delta z} \right] \quad \ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots \quad (10)$$

Sekarang untuk persamaan (10) dibagi oleh $\Delta x \, \Delta y \, \Delta z$ akan-diperoleh bentuk :

$$\frac{\partial \rho}{\partial t} = \left\{ \frac{(\rho V_y)\big|_y - (\rho V_y)\big|_{y+\Delta y}}{\Delta y} \right\} + \left\{ \frac{(\rho V_z)\big|_z - (\rho V_z)\big|_{z+\Delta z}}{\Delta z} \right\}$$

$$+ \left\{ \frac{(\rho V_x)\big|_x - (\rho V_x)\big|_{x+\Delta x}}{\Delta x} \right\}$$

Tiap-tiap suku yang berselisih diambil harga limitnya untuk -

Δx mendekati 0

Δy mendekati 0

Δz mendekati 0

Sehingga :

$$\frac{\partial \rho}{\partial t} = \lim_{\Delta y \to 0} \left\{ \frac{(\rho V_y)\big|_y - (\rho V_y)\big|_{y+\Delta y}}{\Delta y} \right\} +$$

$$\lim_{\Delta z \to 0} \left\{ \frac{(\rho V_z)\big|_z - (\rho V_z)\big|_{z+\Delta z}}{\Delta z} \right\} +$$

$$\lim_{\Delta x \to 0} \left\{ \frac{(\rho V_x)\big|_x - (\rho V_x)\big|_{x+\Delta x}}{\Delta x} \right\}$$

atau :

$$\frac{\partial \rho}{\partial t} = -\frac{\partial (\rho V_x)}{\partial x} - \frac{\partial (\rho V_y)}{\partial y} - \frac{\partial (\rho V_z)}{\partial z}$$

atau :

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho V_x)}{\partial x} + \frac{\partial (\rho V_y)}{\partial y} + \frac{\partial (\rho V_z)}{\partial z} = 0$$

Dalam mencari Model Matematik mengabaikan mengabaikan reaksi - kimia yang terjadi, Maka :

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho V_x)}{\partial x} + \frac{(\rho V_y)}{\partial y} + \frac{(\rho V_z)}{\partial z} = 0$$

.... (11)

dimana :

t = waktu

ρ = density

V_x , V_y , V_z = kecepatan ke arah x , y , z

3.2 KONTINUITAS DALAM LINGKARAN (SILINDER)

Untuk mendapatkan Model Matematik dalam bentuk Lingkaran (Sylinder),diambil koordinat polar. Sedangkan hubungan antara koordinat polar dan koordinat kartesius sbb :

$$x = r \cos \emptyset$$

$$y = r \sin \emptyset$$

Sehingga Model Matematik dari persamaan (11)

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho V_x)}{\partial x} + \frac{\partial(\rho V_y)}{\partial y} + \frac{\partial(\rho V_z)}{\partial z} = 0$$

dapat dirubah dalam bentuk polar dengan menggunakan hubungan diatas. Sebelumnya kita bahas terlebih dahulu substitusinya.

$$dx = \cos \emptyset \, dr - r \sin \emptyset \, d\emptyset$$

$$dy = \sin \emptyset \, dr + r \cos \emptyset \, d\emptyset$$

Untuk mendapatkan harga dari dr dan d$\emptyset$ dicari dengan cara determinan Jacobi :

$$dr = \frac{\begin{vmatrix} dx & -r \sin \emptyset \\ dy & r \cos \emptyset \end{vmatrix}}{\begin{vmatrix} \cos \emptyset & -r \sin \emptyset \\ \sin \emptyset & r \cos \emptyset \end{vmatrix}}$$

$$= \frac{r\,(\cos \emptyset \, dx + \sin \emptyset \, dy}{r.1}$$

Sedang untuk $d\phi$:

$$d\phi = \frac{\begin{vmatrix} \cos\phi & dx \\ \sin\phi & dy \end{vmatrix}}{\begin{vmatrix} \cos\phi & -r\sin\phi \\ \sin\phi & r\cos\phi \end{vmatrix}}$$

$$= \frac{-\sin\phi\, dx + \cos\phi\, dy}{r.1}$$

$$dr = \frac{\partial r}{\partial x} dx + \frac{\partial r}{\partial y} dy = \cos\phi\, dx + \sin\phi\, dy$$

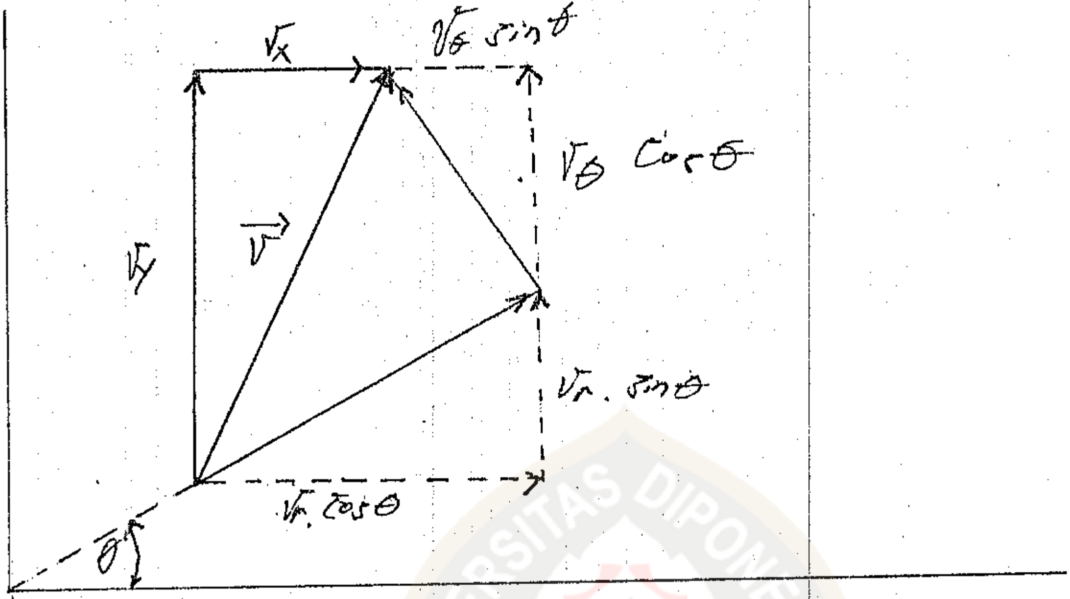
$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy = \frac{-\sin\phi}{r} dx + \frac{\cos\phi}{r} dy$$

Sehingga diperoleh :

$$\frac{\partial r}{\partial x} = \cos\phi \quad \text{dan} \quad \frac{\partial r}{\partial y} = \sin\phi$$

$$\frac{\partial \phi}{\partial x} = \frac{-\sin\phi}{r} \quad \text{dan} \quad \frac{\partial \phi}{\partial y} = \frac{\cos\phi}{r}$$

Untuk kecepatan vektor V diuraikan pada arah x dan y seperti - tampak pada gambar :



$$V_x = V_r \,\text{Cos}\,\emptyset - V_\emptyset \,\text{Sin}\,\emptyset$$

$$V_y = V_\emptyset \,\text{Cos}\,\emptyset + V_r \,\text{Sin}\,\emptyset$$

Perhatikan untuk :

$$\frac{\partial(\rho V_x)}{\partial x} = \frac{\partial(\rho V_x)}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial(\rho V_x)}{\partial \emptyset}\frac{\partial \emptyset}{\partial x} + \frac{\partial(\rho V_x)}{\partial z}\frac{\partial z}{\partial x}$$

$$= \frac{\partial}{\partial r}\left[V_r \,\text{Cos}\,\emptyset - V_\emptyset \,\text{Sin}\,\emptyset\right] . \text{Cos}\,\emptyset +$$

$$\frac{\partial}{\partial \emptyset}\left[V_r \,\text{Sin}\,\emptyset + V_\emptyset \,\text{Cos}\,\emptyset\right] . -\frac{\text{Sin}\,\emptyset}{r}$$

$$\frac{\partial(\rho V_y)}{\partial y} = \frac{\partial(\rho V_y)}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial(\rho V_y)}{\partial \emptyset}\frac{\partial \emptyset}{\partial y} + \frac{\partial(\rho V_y)}{\partial z}\frac{\partial z}{\partial y}$$

$$= \frac{\partial}{\partial r}\left[\rho V_r \,\text{Sin}\,\emptyset + \rho V_\emptyset \,\text{Cos}\,\emptyset\right] . \text{Cos}\,\emptyset +$$

$$\frac{\partial}{\partial \emptyset}\left[\rho V_r \,\text{Sin}\,\emptyset + \rho V_\emptyset \,\text{Cos}\,\emptyset\right] . \frac{\text{Cos}\,\emptyset}{r}$$

Maka :

$$\frac{\partial(\rho V_x)}{\partial x} + \frac{\partial(\rho V_y)}{\partial y} = \frac{\partial}{\partial r}\left[\rho V_r \text{ Cos } \phi - \rho V_\phi \text{ Sin } \phi\right].\text{Cos } \phi$$

$$+\frac{\partial}{\partial \phi}\left[\rho V_r \text{ Sin } \phi + \rho V_\phi \text{ Cos } \phi\right] . -\frac{\text{Sin } \phi}{r} + \frac{\partial}{\partial r}\left[\rho V_r \text{ Sin}/\phi\right.$$

$$\left. + \rho V_\phi \text{ Cos } \phi\right]. \text{ Sin } \phi + \frac{\partial}{\partial \phi}\left[\rho V_r \text{ Sin } \phi + \rho V_\phi \text{ Cos } \phi\right].$$

$$\frac{\text{Cos } \phi}{r} = \frac{1}{r}\frac{\partial}{\partial r}(r \rho V_r) + \frac{1}{r}\frac{\partial}{\partial \phi}(\rho V_\phi)$$

Disubsitusikan pada persamaan (11) didapat :

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r \rho V_r) + \frac{1}{r}\frac{\partial}{\partial \phi}(\rho V_\phi) = 0 \quad \ldots\ldots (12)$$

dimana :

ρ = density masa

V_r = kecepatan ke arah r

V_ϕ = kecepatan ke arah ϕ

r = jari-jari

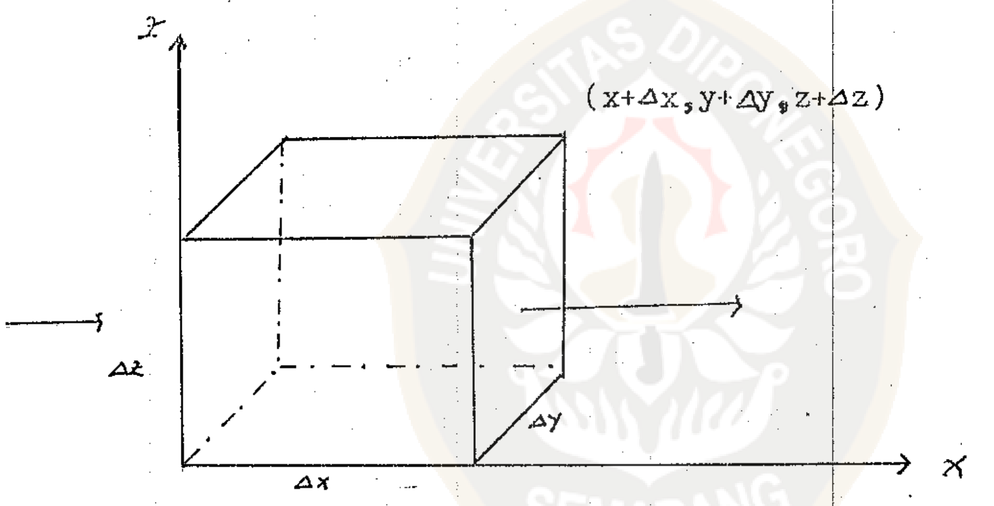
t = waktu

3.3 GERAK DARI ALIRAN FLUIDA

Gerak dari aliran fluida merupakan kelanjutan dari Kontinuitas fluida. Untuk mendapatkan Model Persamaan Differensial dari Gerak Aliran diambil asumsi asumsi sbb :

- Arah perpindahan momentum dari fluida
- Perpindahan gaya
- Laju alir momentum

Dengan mengambil elemen kecil $\Delta x, \Delta y, \Delta z$ sebagai sisi dari volume yang dilewati aliran fluida,seperti tampak pd gambar :



Karena keadaan fluida bergerak ,maka fluida yang masuk akan segera kelar dari volume.

Bila fluida masuk pd x=x,maka akan keluar pd $x=x+\Delta x$

Bila fluida masuk pd y=y maka akan keluar pd $y=y+\Delta y$

Bila fluida masuk pd z=z maka akan keluar pd $z=z+\Delta z$

Pada aliran ada perpindahan Momentum dari aliran dimana perpindahan momentum terpisah menjadi dua(2) yaitu ;

- Perpindahan Momentum
- Perpindahan Molekuler

Peristiwa perpindahan ini juga mengalami perpindahan arah sesuai dengan koordinatnya.

A. Pada arah x melalui bidang $\Delta z \Delta y$ berpindah secara Konveksi

Luas bidang $= \Delta z \cdot \Delta y$

Laju alir Momentum yg masuk $= \Delta z\, \Delta y\, \rho V_x \cdot V_{masuk\ pd\ x=x}$

Laju alir Moementum yg keluar $= \Delta z\, \Delta y\, \rho V_x \cdot V_{keluar\ pd\ x=x+\Delta x}$

Laju Akumulasi Momentum $= \Delta z\, \Delta y \cdot \Delta x \cdot \dfrac{\partial V_x}{\partial t}$

Dengan menggunakan persamaan (3) maka

Laju Akumulasi Momentum = Laju alir Momentum masuk - Laju alir Momentum keluar

atau :

$$\Delta z \Delta y \Delta x \frac{\partial V_x}{\partial t} = \Delta z \Delta y \rho V_x\, V_{masuk\ pd\ x=x} - \Delta z \Delta y \rho V_x\, V_{klr\, x+\Delta}$$

Sedang perpindahan Molekuler pd arah x :

Luas bidang : $\Delta z \cdot \Delta y$

Laju Akumulasi Momentum = Laju alir momentum masuk - Laju alir Momentum keluar.

Laju Akumulasi $= \Delta z\, \Delta y\, \tau_{x\ masuk\ pd\ x=x} - \Delta z\, \Delta y\, \tau_{x\ keluar\ pd\ x=x+\Delta x}$

Untuk perpindahan tekanan dan grafitasi :

Akumulasi tekanan $= \Delta z \Delta y \cdot p|_{masuk\ pd\ x=x} - \Delta z \Delta y \cdot p|_{keluar\ x=x+\Delta x}$

$+ \quad g_x \Delta x\, \Delta y \Delta z$

Untuk Laju Akumulasi Momentum arah x menimbulkan perpindahan Momentum ke arah y.

Luas bidang $= \Delta x \Delta z$

Konveksi : $\Delta x \Delta z \rho V_x\, V_{masuk\ pd\ y=y} - \Delta x \Delta z \rho V_x\, V_{keluar\ pd\ y=y+\Delta y}$

Molekuler : $\Delta x \Delta z\, \tau_{x\ masuk\ pd\ y=y} - \Delta x \Delta z\, \tau_{x\ keluar\ pd\ y=y+\Delta y}$

Untuk Laju Akumulasi Momentum arah x menimbul-kan perpindahan Momentumke arah z.

Luas bidang $= \Delta x \Delta y$

Laju Akumulasiberpindah secara :

Konveksi : $\Delta x \Delta y \; \rho V_x V_{\text{masuk pd } z=z} - \Delta x \Delta y \; \rho V_x V_{\text{keluar } z=z+\Delta z}$

Molekuler : $\Delta x \Delta y \; \tau_{x \text{ masuk pd } z=z} - \Delta x \Delta y \; \tau_{x \text{ keluar pd } z=z+\Delta z}$

Jadi didapat Laju Akumulasi Momentum keseluruhan pada arah x :

$$\begin{aligned}
\Delta x \, \Delta y \Delta z \frac{\partial V_x}{\partial t} &= \Delta y \Delta z \left(\rho V_x V_x\big|_x - \rho V_x V_x\big|_{x+\Delta x} \right) \\
&+ \Delta x \Delta z \left(\rho V_x V_y\big|_y - \rho V_x V_y\big|_{y+\Delta y} \right) \\
&+ \Delta x \Delta y \left(\rho V_x V_z\big|_z - \rho V_x V_z\big|_{z+\Delta z} \right) \\
&+ \Delta y \Delta z \left(\tau_{xx}\big|_x - \tau_{xx}\big|_{x+\Delta x} \right) \\
&+ \Delta x \Delta z \left(\tau_{xy}\big|_y - \tau_{xy}\big|_{y+\Delta y} \right) \\
&+ \Delta x \Delta y \left(\tau_{xz}\big|_z - \tau_{xz}\big|_{z+\Delta z} \right) \\
&+ \Delta y \Delta z \left(p\big|_x - p\big|_{x+\Delta x} \right)
\end{aligned}$$

Setiap suku persamaan ini dibagi oleh $\Delta x \ \Delta y \ \Delta z$ maka :

$$\frac{\partial \rho V_x}{\partial t} = \frac{(\rho V_x V_x|_x - \rho V_x V_x|_{x+\Delta x})}{\Delta x} + \frac{(\rho V_x V_y|_y - \rho V_x V_y|_{y+\Delta y})}{\Delta y}$$

$$+ \frac{(\rho V_x V_z|_z - \rho V_x V_z|_{z+\Delta z})}{\Delta z} \quad \frac{(\tau_{xx}|_x - \tau_{xx}|_{x+\Delta x})}{\Delta x}$$

$$\frac{(\tau_{xy}|_y - \tau_{xy}|_{y+\Delta y})}{\Delta y} \quad \frac{(\tau_{xz}|_z - \tau_{xz}|_{z+\Delta z})}{\Delta z}$$

$$\frac{(p|_x - p|_{x+\Delta x})}{\Delta x} + \rho g_x$$

Karena harga-harga $\Delta x \ \Delta y \ \Delta z$ kecil sekali maka diambil harga limitnya untuk $\Delta x \to 0$, $\Delta y \to 0$ dan $\Delta z \to 0$ sehingga :

$$= \lim_{\Delta x \to 0} \frac{(\rho V_x V_x|_x - \rho V_x V_x|_{x+\Delta x})}{\Delta x}$$

$$+ \lim_{\Delta y \to 0} \frac{(\rho V_x V_y|_y - V_x V_y|_{y+\Delta y})}{\Delta y}$$

$$+ \lim_{\Delta z \to 0} \frac{(\rho V_x V_z|_z - \rho V_x V_z|_{z+\Delta z})}{\Delta z}$$

$$+ \lim_{\Delta x \to 0} \frac{(\tau_{xx}|_x - \tau_{xx}|_{x+\Delta x})}{\Delta x}$$

$$+ \lim_{\Delta y \to 0} \frac{(\tau_{xy}|_y - \tau_{xy}|_{y+\Delta y})}{\Delta y}$$

$$+ \lim_{\Delta z \to 0} \frac{(\tau|_{xz\, z} - \tau_{xz}|_{z+\Delta z})}{\Delta z}$$

$$+ \lim_{\Delta x \to 0} \frac{(p|_{x} - p|_{x+\Delta x})}{\Delta x} + \rho g_x$$

maka ;

$$\frac{\partial \rho V_x}{\partial t} = \frac{\partial \rho V_x V_x}{\partial x} - \frac{\partial \rho V_x V_y}{\partial y} - \frac{\partial \rho V_x V_z}{\partial z} - \frac{\partial \tau_{xx}}{\partial x} - \frac{\partial \tau_{xy}}{\partial y}$$

$$- \frac{\partial \tau_{xy}}{\partial y} - \frac{\partial \tau_{xz}}{\partial z} - \frac{\partial p}{\partial x} + \rho g_x$$

atau ;

$$\frac{\partial \rho V_x}{\partial t} + \frac{\partial \rho V_x V_x}{\partial x} + \frac{\partial \rho V_x V_y}{\partial y} + \frac{\partial \rho V_x V_z}{\partial z} = -\left(\frac{\partial \tau_{xx}}{\partial x} + \right.$$

$$\left. \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} \right) - \frac{\partial p}{\partial x} + \rho g_x$$

untuk ruas kiri dideffrensialkan

$$\rho \left(\frac{\partial V_x}{\partial t} + V_x \frac{\partial V_x}{\partial x} + V_y \frac{\partial V_x}{\partial y} + V_z \frac{\partial V_x}{\partial z} \right) + V_x \left(\frac{\partial \rho}{\partial t} + \frac{\partial \rho V_y}{\partial y} \right.$$

$$\left. + \frac{\partial \rho V_x}{\partial x} + \frac{\partial \rho V_z}{\partial z} \right) = -\left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} \right)$$

$$- \frac{\partial p}{\partial x} + \rho g_x$$

Menurut persamaan Kontinuitas :

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho V_x}{\partial x} + \frac{\partial \rho V_y}{\partial y} + \frac{\partial \rho V_z}{\partial z} = 0$$

Sehingga persamaan berubah menjadi :

$$\rho \left(\frac{\partial V_x}{\partial t} + V_x \frac{\partial V_x}{\partial x} + V_y \frac{\partial V_x}{\partial y} + V_z \frac{\partial V_x}{\partial z} \right) =$$

$$- \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} \right) + \left(- \frac{\partial p}{\partial x} \right) + \rho g_x$$

dimana :

index pertama menyatakan arah perpindahan

index kedua menyatakan arah vektor

V = kecepatan fluida

p = gaya tekan

g = gaya gravitasi

τ = tegangan geser

B. Pada arah : y melalui bidang $\Delta x \Delta z$

Luas bidang $= \Delta x . \Delta z$

Berpindah secara Konveksi :

Laju alir Momentum yg masuk bidang $= \Delta x \Delta z \rho V_y . V_{\text{masuk pd } y=y}$

Laju alir Momentum yg keluar bidang $= \Delta x \Delta z \rho V_y . V_{\text{keluar pd } y=y+\Delta y}$

Laju Akumulasi Momentum $= \Delta z \Delta x \Delta y \dfrac{\partial V_y}{\partial t}$

Dengan menggunakan persamaan (5), maka :

$$\Delta z \Delta x \Delta y \frac{\partial V_y}{\partial t} = \Delta x \Delta z \rho V_y V_{\text{masuk pd } y=y} - \Delta x \Delta z \rho V_y V_{\text{keluar } y=y+\Delta y}$$

Berpindah secara molekuler :

Laju Akumulasi Momentum $= \Delta x \Delta z \tau_{y \text{ masuk pd } y=y} - \Delta x \Delta z \tau_{y \text{ keluar } y=y+\Delta y}$

Pemindahan Momentum oleh tekanan : $(p|_y - p|_{y+\Delta y})$

Pemindahan grafitasi : $g_y \Delta x \Delta y \Delta z$

Laju Akumulasi Momentum arah y menimbulkan perindahan Momentum ke arah z

Luas bidang $= \Delta x \Delta y$

Perpindahan secara :

Konveksi : $\Delta x \Delta y \rho V_y V_{\text{msk pd } z=z} - \Delta x \Delta y \rho V_y V_{\text{klr pd } z=z+\Delta z}$

Molekuler : $\Delta x \Delta y \tau_{y \text{ msk pd } z=z} - \Delta x \Delta y \tau_{y \text{ klr pd } z=z+\Delta z}$

Laju Akumulasi Momentum ke arah y menimbulkan perpindahan Momentum ke arah x

Luas bidang $= \Delta y \Delta z$

Perpindahan secara :

Konveksi : $\Delta y \Delta z \rho V_y V_{\text{msk pd } x=x} - \Delta y \Delta z \rho V_y V_{\text{klr pd } x=x+\Delta x}$

Molekuler : $\Delta y \Delta z \tau_{y \text{msk pd } x=x} - \Delta y \Delta z \tau_{y \text{ klr pd } x=x+\Delta x}$

Jadi jumlah Laju Akumulasi Momentum ke arah y :

$$\Delta x \Delta y \Delta z \frac{\partial V_y}{\partial t} = \Delta x \Delta z \rho V_y V_y - \Delta x \Delta z \rho V_y V_y|_{y+\Delta y}$$
$$+ \Delta y \Delta z \rho V_y V_x|_x - \Delta y \Delta z \rho V_y V_x|_{x+\Delta x}$$
$$+ \Delta x \Delta y \rho V_y V_z|_z - \Delta x \Delta y \rho V_y V_z|_{z+\Delta z}$$
$$+ \Delta x \Delta z\, \tau_{yy}|_y - \Delta x \Delta z\, \tau_{yy}|_{y+\Delta y}$$
$$+ \Delta y \Delta z\, \tau_{yx}|_x - \Delta y \Delta z\, \tau_{yx}|_{x+\Delta x}$$
$$+ \Delta x \Delta y\, \tau_{yz}|_z - \Delta x \Delta y\, \tau_{yz}|_{z+\Delta z}$$
$$+ \Delta x \Delta z\, (p|_y - p|_{y+\Delta y}) + \Delta x \Delta y \Delta z\, g_y$$

Setiap suku dari persamaan ini dibagi dengan $\Delta x \ \Delta y \ \Delta z$,maka :

$$\frac{\partial \rho V_y}{\partial t} = \frac{(\rho V_x V_y|_x - \rho V_x V_y|_{x+\Delta x})}{\Delta x} + \frac{(\rho V_y V_y|_y - \rho V_y V_y|_{y+\Delta y})}{\Delta y}$$
$$+ \frac{(\rho V_z V_y|_z - \rho V_z V_y|_{z+\Delta z})}{\Delta z} + \frac{(\tau_{xy}|_x - \tau_{xy}|_{x+\Delta x})}{\Delta x}$$
$$+ \frac{(\tau_{yy}|_y - \tau_{yy}|_{y+\Delta y})}{\Delta y} + \frac{(\tau_{zy}|_z - \tau_{zy}|_{z+\Delta z})}{\Delta z}$$
$$+ \frac{(p|_y - p|_{y+\Delta y})}{\Delta y} + \rho g_y$$

Karena harga-harga dari $\Delta x, \Delta y, \Delta z$ kecil sekali maka diambil harga limitnya untuk $\Delta x, \Delta y, \Delta z$ mendekati nol ,sehingga :

$$= \lim_{x \to 0} \frac{(\rho V_x V_y|_x - \rho V_x V_y|_{x+\Delta x})}{\Delta x} + \lim_{y \to 0} \frac{(\rho V_y V_y|_y -}{\Delta y}$$

$$\frac{\rho V_y V_y|_{y+\Delta y})}{} + \lim_{z \to 0} \frac{(\rho V_z V_y|_z - \rho V_z V_y|_{z+\Delta z})}{\Delta z}$$

$$+ \lim_{\Delta x \to 0} \frac{(\tau_{xy}|_x - \tau_{xy}|_{x+\Delta x})}{\Delta x}$$

$$+ \lim_{\Delta y \to 0} \frac{(\tau_{yy}|_y - \tau_{yy}|_{y+\Delta y})}{\Delta y}$$

$$+ \lim_{\Delta z \to 0} \frac{(\tau_{zy}|_z - \tau_{zy}|_{y+\Delta y})}{\Delta z}$$

$$+ \lim_{\Delta y \to 0} \frac{(p|_y - p|_{y+\Delta y})}{\Delta y} + \rho g_y$$

maka :

$$\frac{\partial \rho V_y}{\partial t} = -\left(\frac{\partial \rho V_x V_y}{\partial x} + \frac{\partial \rho V_y V_y}{\partial y} + \frac{\partial \rho V_z V_y}{\partial z}\right) - \frac{\partial \tau_{xy}}{\partial x} - \frac{\partial \tau_{yy}}{\partial y}$$

$$- \frac{\partial \tau_{zy}}{\partial z} - \frac{\partial p}{\partial y} + \rho g_y$$

atau :

$$\frac{\partial \rho V_y}{\partial t} + \frac{\partial \rho V_x V_y}{\partial x} + \frac{\partial \rho V_y V_y}{\partial y} + \frac{\partial \rho V_z V_y}{\partial z} = \frac{\partial \tau_{xy}}{\partial x} - \frac{\partial \tau_{yy}}{\partial y} -$$

$$\frac{\partial \tau_{zy}}{\partial z} - \frac{\partial p}{\partial y} + \rho g_y$$

atau :

$$\frac{\partial \rho V_y}{\partial t} + \frac{\partial \rho V_x V_y}{\partial x} + \frac{\partial \rho V_y V_y}{\partial y} + \frac{\partial \rho V_z V_y}{\partial z} = -\frac{\partial \tau_{xy}}{\partial x} - \frac{\partial \tau_{yy}}{\partial y} - \frac{\partial \tau_{zy}}{\partial z}$$

$$- \frac{\partial p}{\partial y} + g_y$$

Untuk ruas kanan dideffrensialkan sehingga didapat :

$$\rho\left(\frac{\partial V_y}{\partial t} + V_x \frac{\partial V_y}{\partial x} + V_y \frac{\partial V_y}{\partial y} + V_z \frac{\partial V_y}{\partial z}\right) + V_y\left(\frac{\partial \rho}{\partial} + \frac{\partial \rho V_x}{\partial x} + \frac{\partial \rho V_y}{\partial y} + \frac{\partial \rho V_z}{\partial z}\right) = -\left\{\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right\} - \frac{\partial p}{\partial y} + \rho g_y$$

Sesuai dengan persamaan Kontinuitas :

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho V_x}{\partial x} + \frac{\partial \rho V_y}{\partial y} + \frac{\partial \rho V_z}{\partial z} = 0$$

Sehingga didapat persamaan :

$$\rho\left(\frac{\partial V_y}{\partial t} + V_x \frac{\partial V_y}{\partial x} + V_y \frac{\partial V_y}{\partial y} + V_z \frac{\partial V_y}{\partial z}\right) = -\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} - \frac{\partial p}{\partial y} + \rho g_y$$

dimana :

index pertama menyatakan arah perpindahan

index kedua menyatakan arah vektor

V = kecepatan fluida

p = gaya tekan

g = gaya gravitasi

τ = tegangan geser

Pada arah z :

$$\Delta y \, \Delta z \, (\rho V_x \, V_z|_x - \rho V_x \, V_z|_{x+\Delta x}) + \Delta x \, \Delta z \, (\rho V_y \, V_z|_y -$$

$$\rho V_y \, V_z|_{y+\Delta y}) + \Delta x \, \Delta y \, (\rho V_z \, V_z|_z - \rho V_z \, V_z|_{z+\Delta z})$$

Demikian juga untuk perpindahan gaya gesernya :

$$\Delta y \, \Delta z \, (\tau_{xz}|_x - \tau_{xz}|_{x+\Delta x}) + \Delta x \, \Delta z \, (\tau_{yz}|_y -$$

$$\tau_{yz}|_{y+\Delta y}) + \Delta x \, \Delta y \, (\tau_{zz}|_z - \tau_{zz}|_{z+\Delta z})$$

Untuk gaya tekan dan gaya gravitasinya :

$$\Delta x \, \Delta y \, (p|_z - p|_{z+\Delta z}) + \rho g_z$$

$\rho V =$ laju akumulasi yang tergantung pada perubahan waktu

jadi ; $$\lim_{t \to 0} \frac{(t + \Delta t)}{\Delta t} = \frac{\partial \rho V}{\partial t}$$

Dengan menggunakan Hukum Kekekalan Masa :

$$\Delta x \, \Delta y \, \Delta z \frac{\partial \rho V}{\partial t} = \Delta y \, \Delta z \, (\rho V_x \, V_z|_x - \rho V_x \, V_z|_{x+\Delta x})$$

$$+ \Delta x \, \Delta z \, (\rho V_y \, V_z|_y - \rho V_y \, V_z|_{y+\Delta y})$$

$$+ \Delta x \, \Delta y \, (\rho V_z \, V_z|_z - \rho V_z \, V_z|_{z+\Delta z})$$

$$+ \Delta y\, \Delta z\, (\tau_{xz}|_x - \tau_{xz}|_{x+\Delta x})$$

$$+ \Delta x\, \Delta z\, (\tau_{yz}|_y - \tau_{yz}|_{y+\Delta y})$$

$$+ \Delta x\, \Delta y\, (\tau_{zz}|_z - \tau_{zz}|_{z+\Delta z})$$

$$+ \Delta x\, \Delta y\, (p|_z - p|_{z+\Delta z}) + \rho g_z$$

Setiap suku dari persamaan ini dibagi dengan $\Delta x\, \Delta y\, \Delta z$ sehingga didapat :

$$\frac{\partial \rho V_z}{\partial t} = \frac{(\rho V_x V_z|_x - \rho V_x V_z|_{x+\Delta x})}{\Delta x}$$

$$+ \frac{(\rho V_y V_z|_y - \rho V_y V_z|_{y+\Delta y})}{\Delta y}$$

$$+ \frac{(\rho V_z V_z|_z - \rho V_z V_z|_{z+\Delta z})}{\Delta z}$$

$$+ \frac{(\tau_{xz}|_x - \tau_{xz}|_{x+\Delta x})}{\Delta x}$$

$$+ \frac{(\tau_{yz}|_y - \tau_{yz}|_{y+\Delta y})}{\Delta y}$$

$$+ \frac{(\tau_{zz}|_z - \tau_{zz}|_{z+\Delta z})}{\Delta z}$$

$$+ \frac{(p|_z - p|_{z+\Delta z})}{\Delta z} + \rho g_z$$

Karena harga-harga dari Δx Δy Δz kecil sekali maka diambil harga limitnya untuk Δx, Δy, Δz mendekati nol, sehingga :

$$\frac{\partial \rho V_z}{\partial t} = \lim_{\Delta x \to 0} \frac{(\rho V_x V_z|_x - \rho V_x V_z|_{x+\Delta x})}{\Delta x}$$

$$+ \lim_{\Delta y \to 0} \frac{(\rho V_y V_z|_y - \rho V_y V_z|_{y+\Delta y})}{\Delta y}$$

$$+ \lim_{\Delta z \to 0} \frac{(\rho V_z V_z|_z - \rho V_z V_z|_{z+\Delta z})}{\Delta z}$$

$$+ \lim_{\Delta x \to 0} \frac{(\tau_{xz}|_x - \tau_{xz}|_{x+\Delta x})}{\Delta x}$$

$$+ \lim_{\Delta y \to 0} \frac{(\tau_{yz}|_y - \tau_{yz}|_{y+\Delta y})}{\Delta y}$$

$$+ \lim_{\Delta z \to 0} \frac{(\tau_{zz}|_z - \tau_{zz}|_{z+\Delta z})}{\Delta z}$$

$$+ \lim_{\Delta z \to 0} \frac{(p|_z - p|_{z+\Delta z})}{\Delta z} + \rho g_z$$

maka :

$$\frac{\partial \rho V_z}{\partial t} = -\left(\frac{\partial \rho V_x V_z}{\partial x} + \frac{\partial \rho V_y V_z}{\partial y} + \frac{\partial \rho V_z V_z}{\partial z}\right)$$

$$- \frac{\partial \tau_{xz}}{\partial x} - \frac{\partial \tau_{yz}}{\partial y} - \frac{\partial \tau_{zz}}{\partial z} - \frac{\partial p}{\partial z} + \rho g_z$$

atau :

$$\frac{\partial \rho V_z}{\partial t} + \frac{\partial \rho V_x V_z}{\partial x} + \frac{\partial \rho V_y V_z}{\partial y} + \frac{\partial \rho V_z V_z}{\partial z} = - \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}$$

$$- \frac{\partial p}{\partial z} + \rho g_z$$

Ruas kanan persamaan ini dideffrensialkan sehingga didapat :

$$\rho \left(\frac{\partial V_z}{\partial t} + V_x \frac{\partial V_z}{\partial x} + V_y \frac{\partial V_z}{\partial y} + V_z \frac{\partial V_z}{\partial z} \right) +$$

$$V_z \left(\frac{\partial \rho}{\partial t} + \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} \right) =$$

$$- \frac{\partial \tau_{xz}}{\partial x} - \frac{\partial \tau_{yz}}{\partial y} - \frac{\partial \tau_{zz}}{\partial z} - \frac{\partial p}{\partial z} + \rho g_z$$

Sesuai dengan persamaan Kontinuitas :

$$\frac{\partial \rho}{\partial t} + \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} = 0$$

Sehingga didapat persamaan :

$$\rho \left(\frac{\partial V_z}{\partial t} + V_x \frac{\partial V_z}{\partial x} + V_y \frac{\partial V_z}{\partial y} + V_z \frac{\partial V_z}{\partial z} \right)$$

$$= - \frac{\partial \tau_{xz}}{\partial x} - \frac{\partial \tau_{yz}}{\partial y} - \frac{\partial \tau_{zz}}{\partial z} - \frac{\partial p}{\partial z} + \rho g_z$$

dimana :

index pertama menyatakan arah perpindahan

index kedua menyatakan arah vektor

V = kecepatan fluida

p = gaya tekan ; g = gaya grafitasi

τ = tegangan geser ; t = waktu

Maka didapat persamaan gerak dalam 3 arah, yaitu :

Pada arah x :

$$\rho\left(\frac{\partial V_x}{\partial t} + V_x\frac{\partial V_x}{\partial x} + V_y\frac{\partial V_x}{\partial y} + V_z\frac{\partial V_x}{\partial z}\right) =$$

$$-\left\{\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right\} - \frac{\partial p}{\partial x} + \rho g_x$$

Pada arah y :

$$\rho\left(\frac{\partial V_y}{\partial} + V_x\frac{\partial V_y}{\partial} + V_y\frac{\partial V_y}{\partial} + V_z\frac{\partial V_y}{\partial}\right) =$$

$$-\left\{\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right\} - \frac{\partial p}{\partial y} + \rho g_y$$

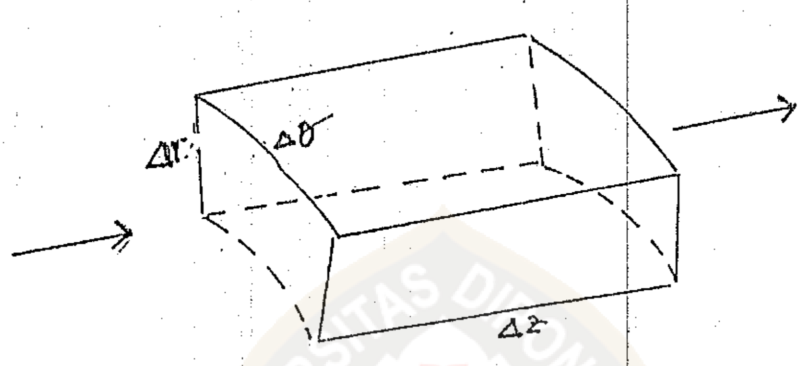
Pada arah z :

$$\rho\left(\frac{\partial V_z}{\partial t} + V_x\frac{\partial V_z}{\partial x} + V_y\frac{\partial V_z}{\partial y} + V_z\frac{\partial V_z}{\partial z}\right) =$$

$$-\left\{\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}\right\} - \frac{\partial p}{\partial z} + \rho g_z$$

3.4 GERAK ALIRAN FLUIDA DALAM PIPA

Sebuah pipa dialiri suatu fluida dengan kecepatan sebesar v untuk mendapatkan Model Matematik dari gerakan aliran ter - sebut diambil pipa dengan sisi-sisi Δr, Δx, $\Delta \emptyset$ yang ber - kedudukan tetap dalam ruang, seperti tampak pada gambar :



Gambar III. yang ditinjau adalah perpindahan momentumnya, untuk menerapkan persamaan (3) diambil pengertian sbb:

$$\text{Luas} = \Delta r \Delta \emptyset$$

Laju Alir Momentum masuk $= \rho V \, . \, V \, . \Delta r \Delta \emptyset$

Laju Alir Momentum keluar $= \rho V \, . \, V \, . \Delta r \Delta \emptyset$

Laju Akumulasi Momentum $= \Delta r \Delta \emptyset \, \rho V \; V_{masuk} - \Delta r \Delta \emptyset \, \rho V \, V_{keluar}$

Dalam setiap fluida mengalir ada dua jenis perpindahan momentum

1. Perpindahan momentum secara molekuler, ditimbulkan karena gaya tarik menarik antar molekul dan arh perpindahan menuju ke-semua arah koordinat.

2. Perpindahan momentum secara konveksi, juga menuju ke se - mua arah koordinat.

Penyebab perpindahan momentum ini adalah aliran masa, maka dise-but perpindahan secara konveksi.

Kecepatan v_z dapat menimbulkan perpindahan momentum ke arah r yang fluksinya dinamakan τ_{rz}, juga ke arah $\emptyset$ fluksinya $\tau_{\emptyset z}$ dan juga ke arah z fluksinya τ_{zz}.

Gaya-gaya yang bekerja pada unsur volume ada dua yakni gaya - tekanan dan gaya gravitasi.

Adanya selisih tekanan antara dua tempat didalam volume akan menimbulkan perubahan kecepatan maupun arahnya dan seterusnya- berakibat pada perubahan fluksi momentum, baik besar dan arahnya

Pada bentuk silinder masih ada satu gaya, yaitu gaya sentrifugal yang ditimbulkan oleh gerakan fluida ke arah Ø, baik secara kon- veksi maupun secara molekuler.

Mula -mula ditinjau komponen ke arah r (Catatan : indeks per - tama menunjukkan arah perpindahan, indeks kedua arah vektor) melalui bidang $\Delta\emptyset\,\Delta z$ berpidah secara :

konveksi ; $(\rho V_r V_r|_r - \rho V_r V_r|_{r+\Delta r})\, r\,\Delta\emptyset\,\Delta z$

molekuler ; $(\tau_{rr}|_r - \tau_{rr}|_{r+\Delta r})\, r\,\Delta\emptyset\,\Delta z$

Melalui bidang $\Delta\emptyset\,\Delta r$ berpindah secara :

konveksi ; $(\rho V_z V_r|_z - \rho V_z V_r|_{z+\Delta z})\, r\,\Delta\emptyset\,\Delta z$

molekuler ; $(\tau_{zr}|_z - \tau_{zr}|_{z+\Delta z})\, r\,\Delta\emptyset\,\Delta z$

Melalui bidang $\Delta r\,\Delta z$ berpindah secara :

konveksi ; $(\rho V_\emptyset V_r|_\emptyset - \rho V_\emptyset V_r|_{\emptyset+\Delta\emptyset})\,\Delta r\,\Delta z$

molekuler ; $(\tau_{\emptyset r}|_\emptyset - \tau_{\emptyset r}|_{\emptyset+\Delta\emptyset})\,\Delta r\,.\,\Delta z$

Perpindahan momentum oleh tekanan dan gaya gravitasi ke arah r

$$(p|_r - p|_{r+\Delta r})\, r\,\Delta\emptyset\,\Delta z + \rho g_r\, r\,\Delta\emptyset\,\Delta r\,\Delta z$$

Gaya sentrifugal ke arah r ialah :

$$\frac{\rho V_\emptyset V_\emptyset}{r}\, r\,\Delta\emptyset\,\Delta r\,\Delta z + \frac{\tau_{\emptyset\emptyset}}{r}\, r\,\Delta\emptyset\,\Delta r\,\Delta z$$

Laju akumulasi momentum kearah r,karena laju ini mengalami perubahan setiap perubahan waktu maka :

$$r\Delta\phi\,\Delta r\,\Delta z \lim_{\Delta t\to 0}\frac{(V_r, t+\Delta t) - (V_r, t)_r}{\Delta t} = r\Delta\phi\,\Delta r\,\Delta z\,\frac{\partial(\rho V_r)}{\partial t}$$

Apabila semua suku laju pindah momentum dijumlahkan sesuai persamaan (3) maka :

$$\begin{aligned}
r\Delta\phi\,\Delta r\,\Delta z\,\frac{\partial(\rho V_r)}{\partial t} = {} & (\rho V_r V_r|_r - \rho V_r V_r|_{r+\Delta r})\, r\Delta\phi\,\Delta z \\
& + (V_\phi V_r|_\phi - V_\phi V_r|_{\phi+\Delta\phi})\,\Delta r\,\Delta z \\
& + (\rho V_z V_r|_z - \rho V_z V_r|_{z+\Delta z})\, r\Delta\phi\,\Delta r \\
& + \frac{V\phi V\phi}{r}\, r\Delta\phi\,\Delta r\,\Delta z \\
& + (\tau_{rr}|_r - \tau_{rr}|_{r+\Delta r})\, r\Delta\phi\,\Delta r \\
& + (\tau_{zr}|_z - \tau_{zr}|_{z+\Delta z})\, r\Delta\phi\,\Delta r \\
& + (\tau_{\phi r}|_\phi - \tau_{\phi r}|_{\phi+\ \phi})\,\Delta r\,\Delta z + \frac{\tau_{\phi\phi}}{r}\, r\Delta\phi\,\Delta r\,\Delta z \\
& + (p|_r - p|_{r+\Delta r})\, r\Delta\phi\,\Delta z + \rho g_r\, r\Delta\phi\,\Delta r\,\Delta z
\end{aligned} \quad \text{.........(13)}$$

Ruas kiri dan kanan persamaan (13)dibagi dengan $r\Delta\phi\,\Delta r\,\Delta z$ maka didapat :

$$\frac{\partial \rho V_r}{\partial t} = \frac{(\rho V_r V_r|_r - \rho V_r V_r|_{r+\Delta r})}{\Delta r} + \frac{(\rho V_\phi V_r|_\phi - \rho V_\phi V_r|_{\phi+\Delta\phi})}{\Delta\phi}$$

$$+ \frac{(\rho V_z V_r|_z - \rho V_z V_r|_{z+\Delta z})}{\Delta z} + \frac{V_\phi V_\phi}{r} + \frac{(\tau_{rr}|_r - \tau_{rr}|_{r+\Delta r})}{\Delta r}$$

$$+ \frac{(\tau_{zr}|_z - \tau_{zr}|_{z+\Delta z})}{\Delta z} + \frac{(\tau_{\phi r}|_\phi - \tau_{\phi r}|_{\phi+\Delta\phi})}{\Delta \phi}$$

$$+ \frac{(p|_r - p|_{r+\Delta r})}{\Delta r} + \rho g_r$$

Kemudian diambil limit dari tiap suku pada waktu $\Delta\phi$, Δr dan Δz mendekati 0,didapat :

$$\frac{\partial \rho V_r}{\partial t} = \lim_{\Delta r \to 0} \frac{(V_r V_r|_r - V_r V_r|_{r+\Delta r})}{\Delta r} + \lim_{\Delta\phi \to 0} \frac{(V_\phi V_r|_\phi - V_\phi V_r|_{\phi+\Delta\phi})}{\Delta\phi}$$

$$+ \lim_{\Delta z \to 0} \frac{(V_z V_r|_z - V_z V_r|_{z+\Delta z})}{\Delta z} + \frac{V_\phi}{r}$$

$$+ \lim_{\Delta r \to 0} \frac{(\tau_{rr}|_r - \tau_{rr}|_{r+\Delta r})}{\Delta r} + \lim_{\Delta z \to 0} \frac{(\tau_{zr}|_z - \tau_{zr}|_{z+\Delta z})}{\Delta z}$$

$$+ \lim_{\Delta\phi \to 0} \frac{(\tau_{\phi r}|_\phi - \tau_{\phi r}|_{\phi+ \phi})}{\Delta\phi} + \frac{\tau_{\phi\phi}}{r} + \lim_{\Delta r \to 0} \frac{(p|_r - p|_{r+\Delta r})}{\Delta r}$$

$$+ \rho g_r .$$

Sehingga :

$$\rho \cdot \frac{\partial V_r}{\partial t} = - \frac{\partial(\rho r V_r V_r)}{\partial r} - \frac{\partial(\rho V_z V_r)}{\partial z} - \frac{\partial(\rho V_\phi V_r)}{\partial \phi} + \frac{\rho V_\phi^2}{r} - \frac{\partial(r\tau_{rr})}{r\partial r}$$

$$- \frac{\partial(\tau_{zr})}{\partial z} - \frac{\partial(\tau_{\phi r})}{r\partial\phi} + \frac{\tau_{\phi\phi}}{r} - \frac{\partial p}{\partial r} + \rho g_r$$

Bila setiap suku yang memuat kecepatan dibawa ke ruas kiri dan dilakukan diferensi,dihasilkan :

$$V_r\left\{\frac{\partial \rho}{\partial t} + \frac{\partial(\rho r V_r)}{r\,\partial r} + \frac{\partial(\rho V_z)}{\partial z} + \frac{\partial(\rho V_\emptyset)}{\partial \emptyset}\right\} + \rho\left(\frac{\partial V_r}{\partial t} + V_r\frac{\partial V_r}{\partial r} + V_z\frac{\partial V_r}{\partial z} + \frac{V_\emptyset}{r}\frac{\partial V_r}{\partial \emptyset} - \frac{V_\emptyset^2}{r}\right) = -\left\{\frac{\partial(r\tau_{rr})}{r\,\partial r} + \frac{\partial \tau_{zr}}{\partial z} + \frac{\partial \tau_{\emptyset r}}{r\,\partial \emptyset} - \frac{\tau_{\emptyset\emptyset}}{r}\right\}$$

$$-\frac{\partial p}{\partial r} + \rho g_r$$

Berdasarkan persamaan Kontinuitas (12) maka kelompok suku pertama dalam ruas kiri besarnya nol, sehingga didapat bentuk :

$$\rho\left(\frac{\partial V_r}{\partial t} + V_r\frac{\partial V_r}{\partial r} + V_z\frac{\partial V_r}{\partial z} + \frac{V_\emptyset}{r}\frac{\partial V_r}{\partial \emptyset} - \frac{V_\emptyset^2}{r}\right) =$$

$$-\left\{\frac{\partial(r\tau_{rr})}{r\,\partial r} + \frac{\partial \tau_{zr}}{\partial z} + \frac{\partial \tau_{\emptyset r}}{r\,\partial \emptyset} - \frac{\tau_{\emptyset\emptyset}}{r}\right\} - \frac{\partial p}{\partial r} + \rho g_r \quad \ldots\ldots (14)$$

Komponen Momentum ke arah Ø.

Melalui bidang $\Delta Ø \Delta z$,berpindah secara :

Konveksi : $(\rho V_r V_Ø|_r - \rho V_r V_Ø|_{r+\Delta r}) r \Delta Ø \Delta z$

Molekuler: $(\tau_{rØ}|_r - \tau_{rØ}|_{r+\Delta r})$

Melalui bidang $\Delta Ø \Delta r$, berpindah secara :

Konveksi : $(\rho V\ V - \rho V\ V) r \Delta Ø \Delta z$

Molekuler : $(\tau_{zØ}|_Ø - \tau_{Øz}|_{Ø+\Delta Ø}) r \Delta Ø \Delta z$

Melalui bidang $\Delta r \Delta z$, berpindah secara :

Konveksi : $(V_Ø V_Ø| - V_Ø V_Ø|_{Ø+\ Ø}) \Delta r \Delta z$

Molekuler : $(\tau_{ØØ}| - \ _{ØØ}|_{Ø+\Delta Ø}) \Delta r \Delta z$

Perpindahan Momentum oleh tekanan dan gaya gravitasi ke arah Ø :

$(p|_Ø - p|_{Ø+\Delta Ø}) \Delta r \Delta z + \rho g_Ø \Delta r \Delta z$

Gaya sentrifugal ke arah Ø ialah :

$$\left(\frac{V_Ø V_Ø}{r} + \frac{\tau_{rØ}}{r}\right) \Delta z \,.\, \Delta r$$

Laju Akumulasi momentum ke arah Ø :

$$r \,.\, \Delta Ø \,.\, \Delta r \,.\, \Delta z \,\frac{\partial(\rho V_Ø)}{\partial t}$$

Berdasarkan persamaan (3), semua suku laju pindah momentum di susun, maka :

$$\Delta r \,.\, \Delta Ø \,.\, \Delta z \,\frac{\partial(\rho V_Ø)}{\partial t} = (\rho V_r V_Ø|_r - \rho V_r V_Ø|_{Ø+\Delta r}) r.\Delta Ø.\Delta z$$

$$+ (\rho V_z . V_Ø|_z - \rho V_z . V_Ø|_{z+\Delta z}) r \Delta Ø \Delta r + (\rho V_Ø V_Ø|_Ø - \rho V_Ø V_Ø$$

$$Ø + \ Ø) r\ z + \frac{V_r V_Ø}{r} . r\ r.\ Ø.\ z + (\ _{rØ}\ r - \ _{rØ}\ r +$$

$$r) r. \Delta Ø \Delta z + (\tau_{zØ}|_z - \tau_{zØ}|_{z+\Delta z}) r \Delta Ø. \Delta r + (\tau_{ØØ}|_Ø$$

$$- \tau_{ØØ}|_{Ø+\Delta Ø}) \Delta z \Delta r + \frac{\tau_{rØ}}{r} r. \Delta r. \Delta Ø. \Delta z$$

dibagi ;

$r \Delta r \Delta Ø \Delta z$

Maka ; $$\frac{\partial(\rho V_\emptyset)}{\partial t} = \frac{(\rho V_r V_\emptyset |r - \rho V_r V_\emptyset |r + \Delta r)}{\Delta t}$$

$$+ \frac{(\rho V_t \cdot V_\emptyset |z - \rho V_t \; V_\emptyset | z + \Delta z)}{\Delta z}$$

$$+ \frac{(\rho V_\emptyset \cdot V_\emptyset |\emptyset - \rho V_\emptyset \cdot V_\emptyset |\emptyset + \Delta\emptyset)}{r . \Delta\emptyset} + \frac{\rho V_r V_\emptyset}{r}$$

$$+ \frac{(\tau_{r\emptyset} |r - \tau_{r\emptyset} |r + \Delta r)}{\Delta r} + \frac{(\tau_{z\emptyset} |z - \tau_{z\emptyset} |z + \Delta z)}{r \Delta\emptyset} + \frac{\tau_{r\emptyset}}{r}$$

Kemudian diambil harga limit pada saat $\Delta r, \Delta z, \Delta\emptyset$ mendekati nol.

$$\frac{\partial\rho}{\partial t} = \lim_{\Delta r \; o} \frac{(\rho V_r V_\emptyset |r - \rho V_r V_\emptyset |r + \Delta r)}{\Delta r}$$

$$+ \lim_{\Delta z \; o} \frac{(\rho V_z V_\emptyset |z - \rho V_r V_\emptyset |z + \Delta z)}{\Delta z}$$

$$+ \lim_{\Delta\emptyset \; o} \frac{(\rho V_\emptyset \cdot V_\emptyset |\emptyset - \rho V_\emptyset \cdot V_\emptyset |\emptyset + \Delta\emptyset)}{r . \Delta\emptyset} + \frac{\rho V_r V_\emptyset}{r}$$

$$+ \lim_{\Delta r} \frac{(\tau_{r\emptyset} |r - \tau_{r\emptyset} |\emptyset + \Delta\emptyset)}{\Delta r}$$

$$+ \lim_{\Delta z \; o} \frac{(\tau_{z\emptyset} |z - \tau_{z\emptyset} |z + \Delta z)}{\Delta z}$$

$$+ \lim_{\Delta\emptyset \; o} \frac{(\tau_{\emptyset\emptyset} |\emptyset - \tau_{\emptyset\emptyset} |\emptyset + \Delta\emptyset)}{\emptyset \Delta\emptyset} + \frac{\tau_{r\emptyset}}{r}$$

Maka ;

$$\frac{\partial V_\emptyset}{\partial t} = - \frac{\partial V_r \cdot V_\emptyset}{\partial r} - \frac{\partial V_z V_\emptyset}{\partial z} - \frac{\partial V_\emptyset V_\emptyset}{r \partial\emptyset} + \frac{\partial V_r V_\emptyset}{\partial\emptyset} - \frac{\partial V_{r\emptyset}}{\partial r}$$

$$- \frac{\partial \tau_{z\emptyset}}{\partial z} - \frac{\partial \tau_{\emptyset\emptyset}}{r \partial\emptyset} + \frac{\partial \tau_{r\emptyset}}{\partial r}$$

Bila semua suku yang memuat kecepatan dibawa ke ruas kiri, maka :

$$\frac{\partial \rho V_\phi}{\partial t} + \frac{\partial \rho V_r}{\partial r} + \frac{\partial \rho V_z V_\phi}{\partial z} + \frac{1}{r}\frac{\partial \rho V_\phi V_\phi}{\partial \phi} - \frac{\partial \rho V_r V_\phi}{r}$$

$$= -\frac{\partial \tau_{r\phi}}{\partial r} - \frac{\tau_{z\phi}}{\partial z} - \frac{\partial \tau_{\phi\phi}}{r\partial \phi} + \frac{\partial \tau_{r\phi}}{\partial r}$$

Untuk ruas kiri dideferensi, maka ;

$$V_\phi \left\{ \frac{\partial \rho}{\partial t} + \frac{\partial V_r}{\partial r} + \frac{\partial V_z}{\partial z} + \frac{1}{r}\frac{\partial V_{\phi r}}{r\partial \phi} \right\} + \rho \left\{ \frac{\partial V_\phi}{\partial t} + V_r \frac{\partial V_\phi}{\partial r} + V_z \frac{\partial V_\phi}{\partial z} + \right.$$

$$\left. \frac{V_\phi}{r}\frac{\partial V_\phi}{\partial \phi} + \frac{V_r V_\phi}{r} \right\} = -\frac{1}{r}\cdot\frac{\partial \tau_{r\phi}}{r} - \frac{\partial \tau_{z\phi}}{z} - \frac{\partial \tau_{\phi\phi}}{r\ \phi} + \frac{V_{r\phi}}{r}$$

$$= -\frac{\partial \tau_{r\phi}}{\partial r} - \frac{\partial \tau_{z\phi}}{\partial z} - \frac{\partial \tau_{\phi\phi}}{r\partial \phi} + \frac{\tau_{r\phi}}{r} \quad \ldots\ldots\ (15)$$

Laju Momentum ke arah z, melalui bidang $\Delta\phi . \Delta z$ berpindah secara ;

Konveksi : $(\rho V_r \cdot V_z |r - \rho V_r \cdot V_z |r + \Delta r)r \cdot \Delta\phi\Delta r$

Molekuler : $(\tau_{rz} |r - \tau_{rz} |r + \Delta r)r\Delta\phi\Delta r$

Melalui bidang $\Delta\phi . \Delta r$ berpindah secara ;

Konveksi : $(\rho V_z \cdot V_z | z - \rho V_z \cdot V_z | z + \Delta z)r\Delta\phi\Delta z$

Molekuler : $(\tau_{zz} | z - \tau_{zz} | z + \Delta z)r\Delta\phi\Delta z$

Melalui bidang $\Delta r\Delta z$ berpindah secara ;

Konveksi ; $(\rho V_\phi V_z | \phi - \rho V_\phi V_z | \phi + \Delta\phi)\Delta r\Delta z$

Molekuler : $(\tau_{\phi z} | \phi - \tau_{\phi z} | \phi + \Delta\phi)\Delta r . \Delta z$

Perpindahan momentum oleh tekanan dan gaya gravitasi ;

$(p|z - p|z + \Delta z)r\Delta r . \Delta\phi\Delta z + g_z\ r\Delta r\Delta\phi\Delta z$

Gaya sentrifugal ke arah r ialah ;

$$\frac{V_r V_z}{r} r\Delta r\Delta\phi\Delta z + \frac{\tau_{\phi z}}{r} r\Delta r\Delta z\Delta\phi$$

Laju Akumulasi momentum ke arah z :

$$r\Delta r\Delta\phi\Delta z\frac{\partial\rho V_z}{\partial t}$$

Sesuai dengan persamaan (3) semua laju pindah momentum disusun, maka didapat bentuk :

$$r\,\Delta r\,\Delta\phi\,\Delta z\,\frac{\partial V_z}{\partial t} = (\rho V_r V_z|_r - \rho V_r V_z|_{r+\Delta r})\, r\,\Delta\phi\,\Delta z + (\rho V_\phi V_z|_\phi$$

$$-\rho V_\phi V_z|_{\phi+\Delta\phi})\Delta z\,\Delta r + (\rho V_z V_z|_z - \rho V_z V_z|_{z+\Delta z})$$

$$+ \frac{V_r V_z}{r}\, r\,\Delta r\,\Delta\phi\,\Delta z + (\tau_{rz}|_r - \tau_{rz}|_{r+\Delta r})$$

$$.\,r\,\Delta\phi\,\Delta z + (\tau_{\phi z}|_\phi + \tau_{\phi z}|_{\phi+\Delta\phi})\,\Delta r\,\Delta z$$

$$+ (\tau_{zz}|_z + \tau_{zz}|_{z+\Delta z})\, r\,\Delta\phi\,\Delta r$$

$$+ (p|_z - p|_{z+\Delta z}) + \rho g_z$$

dibagi oleh $r\,\Delta r\,\Delta\phi\,\Delta z$ akan didapat

$$\frac{\partial\rho V_z}{\partial t} = \frac{(\rho V_r V_z|_r - \rho V_r V_z|_{r+\Delta r})}{\Delta r} + \frac{(\rho V_\phi V_z|_z - \rho V_\phi V_z|_{z+\Delta z})}{r\,\Delta\phi}$$

$$+ \frac{(\rho V_z V_z|_z - \rho V_z V_z|_{z+\Delta z})}{\Delta z} + \frac{V_r V_z}{r} + \frac{(\tau_{rz}|_r - \tau_{rz}|_{r+\Delta r})}{\Delta r}$$

$$+ \frac{(\tau_{\phi z}|_\phi - \tau_{\phi z}|_{\phi+\Delta\phi})}{\Delta\phi} + \frac{(\tau_{zz}|_z - \tau_{zz}|_{z+\Delta z})}{\Delta z}$$

$$+ \frac{(p|_z - p|_{z+\Delta z})}{\Delta z} + \rho g_z$$

Kemudian diambil limit dari tiap suku pada saat $\Delta r\ \Delta\phi$ dan Δz mendekati nol, maka didapat :

$$\frac{\partial \rho V_z}{\partial t} = \lim_{\Delta r \to 0} \frac{(V_r V_z|_r - \rho V_r V_z|_{r+\Delta r})}{\Delta r} + \lim_{\Delta\phi \to 0} \frac{(\rho V_\phi V_z|_\phi - \rho V_\phi V_z|_{\phi+\Delta\phi})}{r\Delta\phi}$$

$$+ \lim_{\Delta z \to 0} \frac{(\rho V_z V_z|_z - \rho V_z V_z|_{z+\Delta z})}{\Delta z} + \lim_{\Delta r \to 0} \frac{(\tau_{rz}|_r - \tau_{rz}|_{r+\Delta r})}{\Delta r}$$

$$+ \lim_{\Delta\phi \to 0} \frac{(\tau_{\phi z}|_\phi - \tau_{\phi z}|_{\phi+\Delta\phi})}{\Delta\phi} + \lim_{\Delta z \to 0} \frac{(\tau_{zz}|_z - \tau_{zz}|_{z+\Delta z})}{\Delta z}$$

$$+ \lim_{\Delta z \to 0} \frac{(p|_z - p|_{z+\Delta z})}{\Delta z} + \frac{\rho V_r V_z}{r} + \rho g_z$$

Bila setiap suku yang memuat kecepatan dibawa ke ruas kiri dan dilakukan diferensi, dihasilkan :

$$V_z \left\{ \frac{\partial \rho}{\partial t} + \frac{\partial(\rho V_r)}{\partial r} + \frac{\partial(\rho V_\phi)}{r\partial\phi} + \frac{\partial(\rho V_z)}{\partial z} \right\} + \rho \left(\frac{\partial V_z}{\partial t} + V_r \frac{\partial V_z}{\partial r} \right.$$

$$\left. + V_z \frac{\partial V_z}{\partial z} + \frac{V_\phi}{r} \frac{\partial V_z}{\partial \phi} \right) = -\left\{ \frac{(\tau_{zr})}{\partial r} + \frac{\partial \tau_{zz}}{\partial z} + \frac{\partial \tau_{\phi z}}{\partial \phi} \right\} \frac{\partial p}{\partial z} + \rho g_z$$

Berdasarkan persamaan Kontinuitas (12), maka kelompok suku pertama dalam ruas kiri besarnya nol, sehingga didapat bentuk :

$$\rho \left(\frac{\partial V_z}{\partial t} + V_r \frac{\partial V_z}{\partial r} + V_z \frac{\partial V_z}{\partial z} + \frac{V_\phi}{r} \frac{\partial V_z}{\partial \phi} \right) = - \frac{\partial (r\tau_{rz})}{\partial r} + \frac{\partial \tau_{zz}}{\partial z}$$

$$+ \frac{\partial \tau_{\phi z}}{r\partial\phi} - \frac{\partial p}{\partial z} + \rho g_z \quad \ldots\ldots\ldots\ldots\ldots \quad (16)$$

Persamaan-persamaan (14),(15),dan (16) adalah komponen ke-arah r,Ø dan z dari persamaan gerak dalam silinder yang dalam simbol vektor dituliskan sbb :

$$\rho \frac{D\vec{V}}{Dt} = -\nabla p - (\nabla \tau) + \rho \vec{g} \qquad \text{..........} \quad (17)$$

dimana :

$$\frac{D\vec{V}}{Dt} = \frac{\partial \vec{V}}{\partial t} + V_r \frac{\partial \vec{V}}{\partial r} + V_z \frac{\partial \vec{V}}{\partial z} + \frac{V_\phi}{r} \frac{\partial \vec{V}}{\partial \phi}$$

Suku-suku persamaan (17) mempunyai arti fisis sbb :

τ = laju pertambhan momentum oleh gesekan

p = gya tekanan

g = gaya gravitasi

= laju pertambhan momentum per satuan volum