BAB II

DIFERENSIAL VEKTOR

II-1. GRADIEN ATAU SLOPE SUATU FUNGSI SKALAR.

Pandang sebuah fungsi skalar $\emptyset$ (u,v)

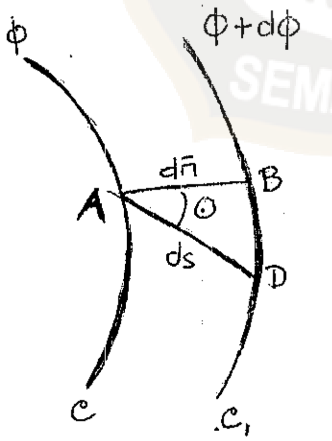
Kurva $\emptyset$ (u,v) = konstan adalah kurva permukaan.

Ditentukan C dan C_1 merupakan kurva permukaan yang ber - urutan sesuai dengan nilai-nilai $\emptyset$ dan $\emptyset + d\emptyset$ dari fungsi, dan $d\emptyset$ adalah positif.

AB merupakan elemen dari trayektori orthogonal dari kurva permukaan yang memotong C dan C_1, $|d\bar{n}|$ merupakan pan - jang elemen tersebut.

AD adalah elemen yang lain, memotong C_1 di D, ds adalah panjang dari AD.

AB merupakan jarak terpendek dari A kekurva C_1 dan mem - punyai arah perubahan $\emptyset$ yang terbesar.



GB. 1.

Gradien dari $\emptyset$ dititik A mempunyai arah AB,

dan besarnya $\dfrac{d\emptyset}{|d\bar{n}|}$

$$\text{Gradien } \emptyset = \bar{\nabla}\emptyset = \frac{d\emptyset}{|d\bar{n}|}$$

$$d\emptyset = (\,\text{grad}\,\phi\,) \cdot |d\bar{n}| = \bar{\nabla}\emptyset \cdot d\bar{n}$$

Karena $\emptyset$ = konstan,

$d\emptyset = 0$

grad $\emptyset \neq 0$, $d\bar{n} \neq 0$

grad $\emptyset \perp d\bar{n}$

Nilai maksimum $d\emptyset$ dihasilkan apabila grad $\emptyset$ dan $d\bar{n}$ berada dalam arah yang sama.

grad $\emptyset$ // $d\bar{n}$

$d\emptyset = \bar{\nabla}\emptyset \cdot |d\bar{n}|$

Harga maksimum dar $d\emptyset$ terjadi pada arah $\bar{\nabla}\emptyset$

Besarnya adalah $|\bar{\nabla}\emptyset|\ |d\bar{n}|$

Gradien $\emptyset$ berada dalam arah perubahan $\emptyset$ yang terbesar.

Hasil diatas dapat disimpulkan sebagai berikut:

a. Vektor grad $\emptyset$ pada sebarang titik, tegak lurus pada garis-garis atau permukaan yang untuknya $\emptyset$ = konstan.

b. Vektor grad $\emptyset$ mempunyai arah perubahan $\emptyset$ yang terbesar.

Jika $\bar{m}$ = unit vektor pada arah AB, orthogonal pada $\emptyset$ = konstan,

$$\bar{\nabla}\emptyset = \frac{d\emptyset}{d\bar{n}}\bar{m} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (\ 2.1.1\ )$$

Pertambahan dari kurva $\emptyset$ pada arah AD:

$$\frac{d\emptyset}{ds} = \frac{d\emptyset}{d\bar{n}} \cdot \frac{d\bar{n}}{ds} = \frac{d\emptyset}{d\bar{n}} \cos\theta$$

θ = sudut inklinasi dari AD terhadap AB.

Kecepatan perubahan $\emptyset$ dalam sebarang arah sepanjang permukaan dapat ditentukan dengan menggunakan vektor satuan dalam arah itu.

Jika $\bar{a}$ = unit vektor pada arah AD,

$$\frac{d\emptyset}{ds} = \bar{a} \cdot \bar{\nabla}\emptyset$$

disebut derivative dari $\emptyset$ pada arah $\bar{a}$.

Jika $d\bar{r}$ adalah vektor awal AD,

$$d\bar{r} = \bar{a}.ds$$

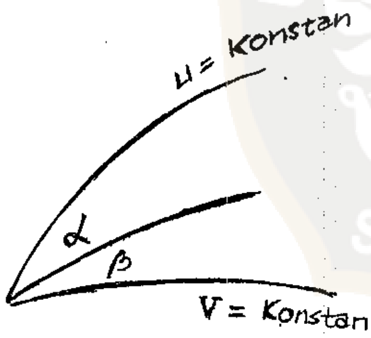
Perubahan $d\emptyset$ dari fungsi karena perubahan $d\bar{r}$ pada permukaan didapatkan dari:

$$d\emptyset = \frac{d\emptyset}{ds} \cdot ds$$

$$= ds\ (\ \bar{a}.\ \bar{\nabla}\emptyset\ )$$

$$d\emptyset = d\bar{r}.\ \bar{\nabla}\emptyset \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\ 2.1.2\ )$$

Gradien $\emptyset$ bebas dari pemilihan parameternya, didapatkan rumusan untuk fungsi dengan koordinat u,v .



Pandang ds sebagai panjang pergeseran $d\bar{r}$ sesuai dengan penambahan du,dv dan δs sebagai panjang pergeseran $\delta\bar{r}$ yang lain sesuai penambahan δu, δv maka:

$$d\bar{r} = \bar{r}_1\ du + \bar{r}_2\ dv$$

$$= \frac{\partial\bar{r}}{\partial u}\ du + \frac{\partial\bar{r}}{\partial v}\ dv$$

$$\delta\bar{r} = \bar{r}_1\ \delta u + \bar{r}_2\ \delta v$$

$$= \frac{\partial\bar{r}}{\partial u}\ \delta u + \frac{\partial\bar{r}}{\partial v}\ \delta v$$

α = sudut antara kurva dan garis parameter u = konstan

β = sudut antara kurva dan garis parameter v = konstan

$$ds.\ \delta s\ \cos\beta = d\bar{r}.\ \delta\bar{r}$$

$$= \bar{r}_1\ \bar{r}_1 du\ \delta u + \bar{r}_1\bar{r}_2\ du\ \delta v + \bar{r}_2\bar{r}_1 dv\ \delta u$$

$$+ \bar{r}_2\bar{r}_2 dv\ \delta v$$

$$ds.\,\delta s \cos\beta = E\, du\, \delta u + F\,(\, du\, \delta v + dv\, \delta u\,) + G\, dv\, \delta v$$

Kedua arah saling tegak lurus jika:

$\cos\beta = 0,\ \beta = \pi/2$

$$E \frac{du}{dv} \frac{\delta u}{\delta v} + F\left(\frac{du}{dv} + \frac{\delta u}{\delta v}\right) + G = 0 \quad \ldots\ldots\ (2.1.3)$$

Persamaan diferensial trayektori orthogonal dari famili kelengkungan diberikan:

$P\,\delta u + Q\,\delta v = 0$

P,Q adalah fungsi dari u,v

$$\frac{\delta u}{\delta v} = -\frac{Q}{P}$$

$$E \frac{du}{dv}\left(-\frac{Q}{P}\right) + F\left\{\frac{du}{dv} + \left(-\frac{Q}{P}\right)\right) + G = 0$$

$$-\,EQ\, du + FP\, du - FQ\, dv + GP\, dv = 0$$

$$EQ\, du - FP\, du + FQ\, dv - GP\, dv = 0$$

$$(\, EQ - FP\,)\, du + (\, FQ - GP\,)\, dv = 0$$

Pandang sekarang $\emptyset\,(u,v) =$ konstan

$$\emptyset_1\, \delta u + \emptyset_2\, \delta v = 0$$

Persamaan diferensial trayektori orthogonal didapatkan dengan mengambil:

$P = \emptyset_1\ ,\ Q = \emptyset_2$

maka

$$(\, EQ_2 - FQ_1\,)\, du + (\, FQ_2 - G\emptyset_1\,)\, dv = 0$$

$$\frac{du}{dv} = \frac{G\emptyset_1 - F\emptyset_2}{E\emptyset_2 - F\emptyset_1}$$

vektor: $\bar{\alpha} = (\, G\emptyset_1 - F\emptyset_2\,)\, \bar{r}_1 + (\, E\emptyset_2 - F\emptyset_1\,)\, \bar{r}_2$

menjadi paralel dengan $\nabla\emptyset$

$$\frac{1}{\sqrt{E}}\, \bar{r}_1 \cdot \bar{\alpha} = (\, G\emptyset_1 - F\emptyset_2\,)\, \frac{E}{\sqrt{E}} + (\, E\emptyset_2 - F\emptyset_1)\frac{F}{\sqrt{E}}$$

$$\frac{1}{\sqrt{E}}\,\bar{r}_1 \cdot \bar{\alpha} = \frac{EG\phi_1 - EF\phi_2}{\sqrt{E}} + \frac{EF\phi_2 - F^2\phi_1}{\sqrt{E}}$$

$$= \frac{(EG - F^2)}{\sqrt{E}}\,\phi_1$$

$$= \frac{H^2}{\sqrt{E}}\,\phi_1$$

$$= H^2 \cdot \frac{1}{\sqrt{E}}\,\frac{\partial\phi}{\partial u}$$

merupakan H^2 kali derivative ϕ dengan arah $\bar{r}_1$.

Jadi gradien ϕ adalah $\dfrac{\bar{\alpha}}{H^2}$

atau :

$$\bar{\nabla}\phi = \frac{(G\phi_1 - F\phi_2)}{H^2}\,\bar{r}_1 + \frac{(E\phi_2 - F\phi_1)}{H^2}\,\bar{r}_2 \qquad \text{------} \quad (2.1.4)$$

II-2. BEBERAPA APLIKASI.

Gradien dari parameter u,v diperoleh dari:

$$\bar{\nabla}u = \frac{G}{H^2}\,\bar{r}_1 - \frac{F}{H^2}\,\bar{r}_2$$

$$\bar{\nabla}v = \frac{E}{H^2}\,\bar{r}_2 - \frac{F}{H^2}\,\bar{r}_1$$

karena:

$$\bar{\nabla}u \times \bar{\nabla}v = \frac{\bar{n}}{H} \quad \text{maka,}$$

$$(\bar{\nabla}u \times \bar{\nabla}v)^2 = \frac{1}{H^2}$$

$$(\bar{\nabla}u)^2 = \frac{G}{H^2} = G\,(\bar{\nabla}u \times \bar{\nabla}v)^2$$

$$\bar{\nabla} u . \bar{\nabla} v = -\frac{F}{H^2} = -F (\bar{\nabla} u \times \bar{\nabla} v)^2$$

$$(\bar{\nabla} v)^2 = \frac{E}{H^2} = E (\bar{\nabla} u \times \bar{\nabla} v)^2$$

Bila kurva parametrik orthogonal, F = 0, maka:

$$(\bar{\nabla} u)^2 = \frac{G}{EG - F^2} = \frac{1}{E}$$

$$(\bar{\nabla} v)^2 = \frac{E}{EG - F^2} , = \frac{1}{G}$$

II-3. DIVERGEN VEKTOR.

Hasil kali skalar operator vektor $\bar{\nabla}$ dengan sebuah fungsi vektor $\bar{\alpha}$ ialah sebuah skalar diferensial invarian yang dinamakan: Divergen $\bar{\alpha}$ atau $\bar{\nabla}.\bar{\alpha}$

$$\text{div } \bar{\alpha} = \bar{\nabla} . \bar{\alpha}$$

$$\text{div } \bar{\alpha} = \frac{1}{H^2} \bar{r}_1 \left(G \frac{\partial \bar{\alpha}}{\partial u} - F \frac{\partial \bar{\alpha}}{\partial v} \right) + \frac{1}{H^2} \bar{r}_2 \left(E \frac{\partial \bar{\alpha}}{\partial v} - F \frac{\partial \bar{\alpha}}{\partial u} \right)$$

Untuk menyatakan pentingnya fungsi divergen, diambil divergensi dari unit normal $\bar{n}$ pada permukaan.

$$\text{div } \bar{n} = \frac{1}{H^2} \bar{r}_1 (G\bar{n}_1 - F\bar{n}_2) + \frac{1}{H^2} \bar{r}_2 (E\bar{n}_2 - F\bar{n}_1)$$

besaran order 2 :

$$L = \bar{n}.\bar{r}_{11} \quad , M = \bar{n}.\bar{r}_{12} \quad , \quad N = \bar{n}.\bar{r}_{22}$$

$$\bar{n} \perp \bar{r}_1 , \quad \bar{n} \perp \bar{r}_2 \quad , \bar{n}.\bar{r}_1 = 0$$

dideferensialkan ke u:

$$\bar{n}_1 \bar{r}_1 + \bar{n} \bar{r}_{11} = 0$$

$\bar{n}_1\bar{r}_1 = -\bar{n}\bar{r}_{11} = -L$

$\bar{n}.\bar{r}_2 = 0$

$\bar{n}_1\bar{r}_2 + \bar{n}\bar{r}_{12} = 0$

$\bar{n}_1\bar{r}_2 = -\bar{n}\bar{r}_{12} = -M$

$\bar{n}.\bar{r}_1 = 0$

dideferensialkan ke v:

$\bar{n}_2\bar{r}_1 + \bar{n}\bar{r}_{21} = 0$

$\bar{n}_2\bar{r}_1 = -\bar{n}\bar{r}_{21} = -M$

$\bar{n}.\bar{r}_2 = 0$

$\bar{n}_2\bar{r}_2 + \bar{n}\bar{r}_{22} = 0$

$\bar{n}_2\bar{r}_2 = -\bar{n}\bar{r}_{22} = -N$

dengan mensubstitusi $\bar{n}_1$ & $\bar{n}_2$ didapatkan:

$$\text{div } \bar{n} = \frac{1}{H^2}(-GL + FM) + \frac{1}{H^2}(-EN + FM)$$

$$= -\frac{1}{H^2}(GL - FM + EN - FM)$$

$$= -\frac{1}{H^2}(EN - 2FM + GL)$$

$$\text{div } \bar{n} = -\partial \quad \ldots\ldots\ldots\ldots\ldots\ldots (2.4.1)$$

∂ = kelengkungan pertama

jadi kelengkungan pertama dari sebuah permukaan adalah negatif dari divergensi normal satuan.

Bila R merupakan sebuah skalar fungsi titik:

$$\bar{n} \perp \bar{r}_1 , \bar{n} \perp \bar{r}_2$$

$$\text{div } R\bar{n} = R \text{ div } \bar{n}$$

$$\text{div } R\bar{n} = -\partial R \quad \ldots\ldots\ldots\ldots \quad (2.4.2)$$

Kelengkungan pertama dari sebuah permukaan minimal adalah sama dengan nol, $\partial = 0$

$$\text{div } \bar{n} = 0$$

$$\text{div } R\bar{n} = 0$$

Divergen suatu vektor dapat dinyatakan dari hubungan beberapa komponen vektor.

$$\bar{\alpha} = P\bar{r}_1 + Q\bar{r}_2 + R\bar{n}$$

$$\text{div } \bar{\alpha} = \text{div } (P\bar{r}_1) + \text{div } (Q\bar{r}_2) + \text{div } (R\bar{n})$$

$$\text{div } P\bar{r}_1 = \frac{1}{H^2}\bar{r}_1 \left\{ G (P_1\bar{r}_1 + P\bar{r}_{11}) - F (P_2\bar{r}_1 + P\bar{r}_{12}) \right\} + \frac{1}{H^2}\bar{r}_2 \left\{ E (P_2\bar{r}_1 + P\bar{r}_{12}) - F (P_1\bar{r}_1 + P\bar{r}_{11}) \right\}$$

$$= \frac{1}{H^2} \left\{ GP_1\bar{r}_1\bar{r}_1 + GP\bar{r}_1\bar{r}_{11} - FP_2\bar{r}_1\bar{r}_1 - FP\bar{r}_1\bar{r}_{12} + EP_2\bar{r}_1\bar{r}_2 + EP\bar{r}_2\bar{r}_{12} - FP_1\bar{r}_1\bar{r}_2 - FP\bar{r}_2\bar{r}_{11} \right\}$$

$$= \frac{1}{H^2} \left\{ EGP_1 + GP\bar{r}_1\bar{r}_{11} - EFP_2 - FP\bar{r}_1\bar{r}_{12} + EFP_2 + EP\bar{r}_2\bar{r}_{12} - F^2P_1 - FP\bar{r}_2\bar{r}_{11} \right\}$$

$$= \frac{1}{H^2} \left\{ (EG - F^2)P_1 + P \left(G \frac{\partial \bar{r}}{\partial u} \frac{\partial^2 \bar{r}}{\partial u^2} + E \frac{\partial \bar{r}}{\partial v} \frac{\partial^2 \bar{r}}{\partial u \partial v} - F \frac{\partial \bar{r}}{\partial u} \frac{\partial^2 \bar{r}}{\partial u \partial v} - F \frac{\partial \bar{r}}{\partial v} \frac{\partial^2 \bar{r}}{\partial u^2} \right. \right.$$

diketahui bahwa:

$\bar{r}_1 = \bar{r}_u$

$\bar{r}_2 = \bar{r}_v$

$E = \bar{r}_u \cdot \bar{r}_u$

$E_u = 2\ \bar{r}_u \cdot \bar{r}_{uu}$

$$\bar{r}_u \cdot \bar{r}_{uu} = \tfrac{1}{2} E_u$$

$$= \tfrac{1}{2} \frac{\partial}{\partial u} (\bar{r}_u)^2$$

$$E_v = 2 \bar{r}_u \cdot \bar{r}_{uv}$$

$$\bar{r}_u \cdot \bar{r}_{uv} = \tfrac{1}{2} E_v$$

$$= \tfrac{1}{2} \frac{\partial}{\partial v} (\bar{r}_u)^2$$

$$F = \bar{r}_u \cdot \bar{r}_v$$

$$F_u = \bar{r}_{uu} \cdot \bar{r}_v + \bar{r}_u \cdot \bar{r}_{uv}$$

$$= \bar{r}_v \cdot \bar{r}_{uu} + \tfrac{1}{2} E_v$$

$$\bar{r}_v \cdot \bar{r}_{uu} = F_u - \tfrac{1}{2} E_v$$

$$= \frac{\partial}{\partial u} (\bar{r}_u \cdot \bar{r}_v) - \tfrac{1}{2} \frac{\partial}{\partial v} (\bar{r}_u)^2$$

$$G = \bar{r}_v \cdot \bar{r}_v$$

$$G_u = 2 \bar{r}_v \cdot \bar{r}_{uv}$$

$$\bar{r}_v \cdot \bar{r}_{uv} = \tfrac{1}{2} G_u$$

$$= \tfrac{1}{2} \frac{\partial}{\partial u} (\bar{r}_v)^2$$

$$\text{div } P\bar{r}_1 = P_1 + \frac{P}{H^2} \left\{ G \frac{\partial}{\partial u} (\bar{r}_u)^2 + E \tfrac{1}{2} \frac{\partial}{\partial u} (\bar{r}_v)^2 - \right.$$

$$\left. - 2F \cdot \tfrac{1}{2} \frac{\partial}{\partial u} (\bar{r}_u \cdot \bar{r}_v)^2 \right\}$$

$$= P_1 + \frac{P}{2H^2} \left\{ E \frac{\partial}{\partial u} (\bar{r}_2)^2 + G \frac{\partial}{\partial u} (\bar{r}_1)^2 - \right.$$

$$\left. - 2F \frac{\partial}{\partial u} (\bar{r}_1 \bar{r}_2) \right\}$$

$$\text{div } P\bar{r}_1 = P_1 + \frac{P}{2H^2}\left\{ E\frac{\partial}{\partial u}(G) + G\frac{\partial}{\partial u}(E) - \right.$$

$$\left. - 2F\frac{\partial}{\partial u}(F)\right\}$$

$$= P_1 + \frac{P}{2H^2}\frac{\partial}{\partial u}(2EG - 2F)$$

$$= P_1 + \frac{P}{H^2}\frac{\partial}{\partial u}(H^2)$$

$$\text{div } P\bar{r}_1 = \frac{1}{H}\frac{\partial}{\partial u}(HP)$$

$$\text{div } (Q\bar{r}_2) = \frac{1}{H^2}\bar{r}_1\left\{ G (Q_1\bar{r}_2 + Q\bar{r}_{12}) - F(Q_2\bar{r}_1 + Q\bar{r}_{22})\right\} +$$

$$+ \frac{1}{H^2}\bar{r}_2\left\{ E (Q_2\bar{r}_2 + Q\bar{r}_{22}) - F(Q_1\bar{r}_2 + Q\bar{r}_{12})\right\}$$

$$= \frac{1}{H^2}\left\{ GQ_1\bar{r}_1\bar{r}_2 + GQ\bar{r}_1\bar{r}_{12} - FQ_2\bar{r}_1\bar{r}_2 - FQ\bar{r}_1\bar{r}_{22} + \right.$$

$$\left. + EQ_2\bar{r}_2\bar{r}_2 + EQ\bar{r}_2\bar{r}_{22} - FQ_1\bar{r}_2\bar{r}_2 - FQ\bar{r}_2\bar{r}_{12} \right\}$$

$$\text{div } (Q\bar{r}_2) = \frac{1}{H^2}(FGQ_1 + GQ\bar{r}_1\bar{r}_{12} - F^2Q_2 - FQ\bar{r}_1\bar{r}_{22} +$$

$$+ EGQ_2 + EQ\bar{r}_2\bar{r}_{22} - FGQ_1 - FQ\bar{r}_2\bar{r}_{12})$$

$$= \frac{1}{H^2}\left\{(EG - F^2) Q_2\right\} + \frac{Q}{H^2}(E\bar{r}_2\bar{r}_{22} +$$

$$+ G\bar{r}_1\bar{r}_{12} - F\bar{r}_1\bar{r}_{22} - F\bar{r}_2\bar{r}_{12})$$

$\bar{r}_1 = \bar{r}_u$

$\bar{r}_2 = \bar{r}_v$

$E = \bar{r}_u \cdot \bar{r}_u$

$E_v = 2\bar{r}_u \cdot \bar{r}_{uv}$

$\bar{r}_u \cdot \bar{r}_{uv} = \frac{1}{2} E_v$

$$\bar{r}_u \cdot \bar{r}_{uv} = \frac{1}{2} \frac{\partial}{\partial v} (\bar{r}_u)^2$$

$$G = \bar{r}_v \cdot \bar{r}_v$$

$$G_v = 2 \, \bar{r}_v \cdot \bar{r}_{vv}$$

$$\bar{r}_v \cdot \bar{r}_{vv} = \frac{1}{2} G_v$$

$$= \frac{1}{2} \frac{\partial}{\partial v} (\bar{r}_v)^2$$

$$G_u = 2\bar{r}_v \cdot \bar{r}_{uv}$$

$$\bar{r}_v \cdot \bar{r}_{uv} = \frac{1}{2} G_u$$

$$= \frac{1}{2} \frac{\partial}{\partial u} (\bar{r}_v)^2$$

$$F = \bar{r}_u \cdot \bar{r}_v$$

$$F_v = \bar{r}_{uv} \cdot \bar{r}_v + \bar{r}_u \cdot \bar{r}_{vv}$$

$$= \frac{1}{2} G_u + \bar{r}_u \cdot \bar{r}_{uv}$$

$$\bar{r}_u \cdot \bar{r}_{vv} = F_v - \frac{1}{2} G_u$$

$$= \frac{\partial}{\partial v} (\bar{r}_u \cdot \bar{r}_v) - \frac{1}{2} \frac{\partial}{\partial u} (\bar{r}_v)^2$$

$$\text{div} (Q\bar{r}_2) = Q_2 + \frac{Q}{H^2} \Big(E \frac{\partial \bar{r}}{\partial v} \frac{\partial^2 \bar{r}}{\partial v^2} + G \frac{\partial \bar{r}}{\partial u} \frac{\partial^2 \bar{r}}{\partial u \partial v} - F \frac{\partial \bar{r}}{\partial u} \frac{\partial^2 \bar{r}}{\partial v^2} -$$

$$- F \frac{\partial \bar{r}}{\partial v} \frac{\partial^2 \bar{r}}{\partial u \partial v} \Big)$$

$$= Q_2 + \frac{Q}{H^2} \Big(E \cdot \frac{1}{2} \frac{\partial}{\partial v} (\bar{r}_v)^2 + G \cdot \frac{1}{2} \frac{\partial}{\partial v} (\bar{r}_u)^2 -$$

$$- F \frac{\partial}{\partial v} (\bar{r}_u \cdot \bar{r}_v) + F \cdot \frac{1}{2} \frac{\partial}{\partial u} (\bar{r}_v)^2 -$$

$$- F \cdot \frac{1}{2} \frac{\partial}{\partial u} (\bar{r}_v)^2$$

$$\text{div}\ (Q\bar{r}_2) = Q_2 + \frac{Q}{2H^2}\left\{ E\frac{\partial}{\partial v}(\bar{r}_2)^2 + G\frac{\partial}{\partial v}(\bar{r}_1)^2 - \right.$$

$$\left. - 2F\frac{\partial}{\partial v}(\bar{r}_1 \cdot \bar{r}_2)\right\}$$

$$= Q_2 + \frac{Q}{2H^2}\left\{ E\frac{\partial}{\partial v}(G) + G\frac{\partial}{\partial v}(E) - \right.$$

$$\left. - 2F\frac{\partial}{\partial v}(F)\right\}$$

$$= Q_2 + \frac{Q}{2H^2}\frac{\partial}{\partial v}(2EG - 2F^2)$$

$$= Q_2 + \frac{Q}{H^2}\frac{\partial}{\partial v}(EG - F^2)$$

$$\text{div}\ (Q\bar{r}_2) = Q_2 + \frac{Q}{H^2}\frac{\partial}{\partial v}(H^2)$$

$$\text{div}\ (Q\bar{r}_2) = \frac{1}{H}\frac{\partial}{\partial v}(HQ)$$

diketahui dalam persamaan (2.4.2):

$$\text{div}\ (R\bar{n}) = -JR$$

Jadi :

$$\text{div}\ \bar{\alpha} = \frac{1}{H}\left\{\frac{\partial}{\partial u}(HP) + \frac{\partial}{\partial v}(HQ)\right\} - JR \qquad \text{......... (2.4.3)}$$

Bila vektor $\bar{\alpha}$ adalah sama dengan gradien fungsi skalar $\emptyset$, yaitu :

$$\bar{\alpha} = \bar{\nabla}\emptyset$$

$$\text{div}\ \bar{\alpha} = \text{div}\ \bar{\nabla}\emptyset$$

$$= \bar{\nabla}\cdot\bar{\nabla}\emptyset$$

$$\text{div}\ \bar{\nabla}\emptyset = \bar{\nabla}^2\emptyset$$

operator $\bar{\nabla}^2$ disebut : operator Laplace atau Laplacian.

dari (2.1.4)

$$\bar{\nabla}\, \emptyset = \frac{(G\emptyset_1 - F\emptyset_2)}{H^2}\, \bar{r}_1 + \frac{(E\emptyset_2 - F\emptyset_1)}{H^2}\, \bar{r}_2 .$$

Eleminasikan nilai-nilai P dan Q dari (2.1.4) pada persamaan (2.4.3)

$$\bar{\nabla}^2 \emptyset = \frac{1}{H} \left\{ \frac{\partial}{\partial u} H \frac{(G\emptyset_1 - F\emptyset_2)}{H^2} + \frac{\partial}{\partial v} H \frac{(E\emptyset_2 - F\emptyset_1)}{H^2} \right\}$$

$$\bar{\nabla}^2 \emptyset = \frac{1}{H} \frac{\partial}{\partial u} \frac{(G\emptyset_1 - F\emptyset_2)}{H} + \frac{1}{H} \frac{\partial}{\partial v} \frac{(E\emptyset_2 - F\emptyset_1)}{H}$$

...... (2.4.4)

Bila kurva parametrik orthogonal :

F = 0

$$\bar{\nabla}^2 \emptyset = \frac{1}{\sqrt{EG}} \left\{ \frac{\partial}{\partial u} \left(\emptyset_1 \sqrt{\frac{G}{E}} \right) + \frac{\partial}{\partial v} \left(\emptyset_2 \sqrt{\frac{E}{G}} \right) \right\}$$

...... (2.4.5)

II-4. <u>KURL SUATU VEKTOR</u>.

Operator $\bar{\nabla}$ yang digunakan untuk suatu fungsi vektor $\bar{\alpha}$ memberikan suatu diferensial invarian vektor, disebut kurl atau Rotasi dari $\bar{\alpha}$.

Didefinisikan :

$$\text{Kurl}\ \bar{\alpha} = \bar{\nabla} \times \bar{\alpha}$$

$$\text{kurl}\ \bar{\alpha} = \frac{1}{H^2}\, \bar{r}_1 \times \left(G \frac{\partial \bar{\alpha}}{\partial u} - F \frac{\partial \bar{\alpha}}{\partial v} \right) + \frac{1}{H^2}\, \bar{r}_2 \times \left(E \frac{\partial \bar{\alpha}}{\partial v} - F \frac{\partial \bar{\alpha}}{\partial u} \right)$$

Ambil kurl dari suatu vektor $R\bar{n}$ normal pada permukaan.

$$\text{kurl } R\bar{n} = \frac{1}{H^2}\bar{r}_1 \times \left\{ G\,(R_1\bar{n} + R\bar{n}_1) - F\,(R_2\bar{n} + R\bar{n}_2) \right\} + \frac{1}{H^2}\bar{r}_2 \times \left\{ E\,(R_2\bar{n} + R\bar{n}_2) - F(R_1\bar{n} + R\bar{n}_1) \right\}$$

$$\text{kurl } R\bar{n} = \bar{\nabla} R \times \bar{n}$$

Jika R = konstan, maka

$$\bar{\nabla} R = 0$$

$$\text{kurl } R\bar{n} = 0$$

$$\text{kurl } \bar{n} = \bar{\nabla} \times \bar{n} = 0$$

$$\text{kurl } \bar{r} = \bar{\nabla} \times \bar{r} = 0$$

Jika :

$$\bar{\alpha} = P\bar{r}_1 + Q\bar{r}_2 + R\bar{n}$$

$$\text{kurl } \bar{\alpha} = \text{kurl } P\bar{r}_1 + \text{kurl } Q\bar{r}_2 + \text{kurl } R\bar{n}$$

$$\text{kurl } P\bar{r}_1 = \frac{1}{H^2}\bar{r}_1 \times \left\{ G\,(P_1\bar{r}_1 + P\bar{r}_{11}) - F\,(P_2\bar{r}_1 + P\bar{r}_{12}) \right\} + \frac{1}{H^2}\bar{r}_2 \times \left\{ E\,(P_2\bar{r}_1 + P\bar{r}_{12}) - F(P_1\bar{r}_1 + P\bar{r}_{11}) \right\}$$

$$= \frac{1}{H^2}\bar{r}_1 \times G\,(P_1\bar{r}_1 + P\bar{r}_{11}) - \frac{1}{H^2}\bar{r}_1 \times F(P_2\bar{r}_1 + P\bar{r}_{12}) + \frac{1}{H^2}\bar{r}_2 \times E(P_2\bar{r}_1 + P\bar{r}_{12}) - \frac{1}{H^2}\bar{r}_2 \times F\,(P_1\bar{r}_1 + P\bar{r}_{11})$$

$$= \frac{1}{H^2}\bar{r}_1 \times \bar{r}_2.\bar{r}_2\,(P_1\bar{r}_1 + P\bar{r}_{11}) - \frac{1}{H^2}\bar{r}_1 \times \bar{r}_1.\bar{r}_2(P_2\bar{r}_1 + P\bar{r}_{12}) + \frac{1}{H^2}\bar{r}_2 \times \bar{r}_1.\bar{r}_1(P_2\bar{r}_1 + P\bar{r}_{12}) - \frac{1}{H^2}\bar{r}_2 \times \bar{r}_1.\bar{r}_2\,(P_1\bar{r}_1 + P\bar{r}_{11})$$

$$= \frac{1}{H}\bar{n}.\bar{r}_2\,(P_1\bar{r}_1 + P\bar{r}_{11}) - \frac{1}{H}\bar{n}.\bar{r}_1(P_2\bar{r}_1 + P\bar{r}_{12}) + \frac{1}{H}\bar{n}.\bar{r}_1(P_2\bar{r}_1 + P\bar{r}_{12}) - \frac{\bar{n}.\bar{r}_2}{H}(P_1\bar{r}_1 + P\bar{r}_{11})$$

$$\text{kurl } P\bar{r}_1 = \frac{\bar{n}.\bar{r}_2\bar{r}_1 P_1}{H} + \frac{\bar{n}.\bar{r}_{11}.\bar{r}_2.P}{H} - \frac{\bar{n}.\bar{r}_1\bar{r}_1 P_2}{H} - \frac{\bar{n}.\bar{r}_1.\bar{r}_{12}P}{H} +$$

$$\frac{\bar{n}.\bar{r}_1 P_2 \bar{r}_1}{H} + \frac{\bar{n}.\bar{r}_1 P.\bar{r}_{12}}{H} - \frac{\bar{n}.\bar{r}_2 P_1 \bar{r}_1}{H} - \frac{\bar{n}.\bar{r}_2 P.\bar{r}_{11}}{H}$$

$$= \frac{1}{H} (\bar{n}.P.\bar{r}_2.\bar{r}_{11} - \bar{n}.P.\bar{r}_1 \bar{r}_{12} + \bar{n}.P.\bar{r}_1 \bar{r}_{12} - \bar{n}.P.\bar{r}_2 \bar{r}_{11})$$

$$\bar{r}_2.\bar{r}_{11} = \frac{\partial}{\partial u} (\bar{r}_1 . \bar{r}_2) - \frac{1}{2} \frac{\partial}{\partial v} (\bar{r}_1 . \bar{r}_1)$$

$$\bar{r}_1 . \bar{r}_{12} = \frac{1}{2} \frac{\partial}{\partial v} (\bar{r}_1 . \bar{r}_1)$$

$$\text{kurl } P\bar{r}_1 = \frac{\bar{n}.P}{H} \left\{ \frac{\partial}{\partial u} (\bar{r}_1 \bar{r}_2) - \frac{1}{2} \frac{\partial}{\partial v} (\bar{r}_1 . \bar{r}_1) - \frac{1}{2} \frac{\partial}{\partial v} (\bar{r}_1 . \bar{r}_1) \right\} +$$

$$\frac{P}{H} (\bar{r}_1 \bar{r}_{12} \bar{n} - \bar{r}_2 \bar{r}_{11} \bar{n})$$

Dengan subtitusi nilai-nilai $\bar{r}_{11}$ dan $\bar{r}_{12}$, dimana $\bar{n}.\bar{r}_{11} = L$, $\bar{n}.\bar{r}_{12} = M$ didapatkan:

$$\text{kurl } P\bar{r}_1 = \frac{1}{H} \left\{ \frac{\partial}{\partial u} (FP) - \frac{\partial}{\partial v} (EP) \right\} \bar{n} + \frac{P}{H} (M\bar{r}_1 - L\bar{r}_2)$$

$$\text{kurl } Q\bar{r}_2 = \frac{1}{H^2} \bar{r}_1 \times \left\{ G (Q_1 \bar{r}_2 + Q.\bar{r}_{12}) - F(Q_2 \bar{r}_2 + Q.\bar{r}_{22}) \right\}$$

$$+ \frac{1}{H^2} \bar{r}_2 \times \left\{ E(Q_2 \bar{r}_2 + Q.\bar{r}_{22}) - F(Q_1 \bar{r}_2 + Q.\bar{r}_{12}) \right\}$$

$$= \frac{1}{H^2} \bar{r}_1 \times G (Q_1 \bar{r}_2 + Q.\bar{r}_{12}) - \frac{1}{H^2} \bar{r}_1 \times F(Q_2 \bar{r}_2 + Q.\bar{r}_{22}) +$$

$$\frac{1}{H^2} \bar{r}_2 \times E (Q_2 \bar{r}_2 + Q.\bar{r}_{22}) - \frac{1}{H^2} \bar{r}_2 \times F (Q_1 \bar{r}_2 +$$

$$Q.\bar{r}_{12})$$

$$= \frac{1}{H^2} \bar{r}_1 \times \bar{r}_2 . \bar{r}_2 (Q_1 \bar{r}_2 + Q.\bar{r}_{12}) - \frac{1}{H^2} \bar{r}_1 \times \bar{r}_1 . \bar{r}_2$$

$$(Q_2 \bar{r}_2 + Q.\bar{r}_{22}) + \frac{1}{H^2} \bar{r}_2 \times \bar{r}_1 . \bar{r}_1 (Q_2 \bar{r}_2 + Q.\bar{r}_{22}) -$$

$$\frac{1}{H^2} \bar{r}_2 \times \bar{r}_1 . \bar{r}_2 (Q_1 \bar{r}_2 + Q.\bar{r}_{12})$$

$$\text{kurl } Q\bar{r}_2 = \frac{1}{H}\,\bar{n}.\bar{r}_2(\,Q_1\bar{r}_2 + Q\bar{r}_{12}\,) - \frac{1}{H}\,\bar{n}.\bar{r}_1(Q_2\bar{r}_2+Q\bar{r}_{22}) +$$

$$+ \frac{1}{H}\,\bar{n}.\bar{r}_1(Q_2\bar{r}_2 + Q\bar{r}_{22}\,) - \frac{1}{H}\,\bar{n}.\bar{r}_2(Q_1\bar{r}_2+Q\bar{r}_{12})$$

$$= \frac{\bar{n}.\bar{r}_2}{H}\,Q_1\bar{r}_2 + \frac{n\bar{r}_2}{H}\,Q\bar{r}_{12} - \frac{\bar{n}\bar{r}_1}{H}\,Q_2\bar{r}_2 - \frac{\bar{n}\bar{r}_1}{H}\,Q\bar{r}_{22} +$$

$$+ \frac{\bar{n}.\bar{r}_1}{H}\,Q_2\bar{r}_2 + \frac{\bar{n}.\bar{r}_1}{H}\,Q\bar{r}_{22} - \frac{\bar{n}.\bar{r}_2}{H}\,Q_1\bar{r}_2 - \frac{\bar{n}.\bar{r}_2}{H}\,Q\bar{r}_{12}$$

$$= \frac{\bar{n}.\bar{r}_2}{H}\,Q\bar{r}_{12} - \frac{\bar{n}.\bar{r}_1}{H}\,Q\bar{r}_{22} + \frac{\bar{n}.\bar{r}_1}{H}\,Q\bar{r}_{22} - \frac{\bar{n}.\bar{r}_2}{H}\,Q\bar{r}_{12}$$

diketahui bahwa:

$$\bar{r}_2.\bar{r}_{12} = \tfrac{1}{2}\,\frac{\partial}{\partial u}(\,\bar{r}_2.\bar{r}_2\,)$$

$$\bar{r}_1.\bar{r}_{22} = \frac{\partial}{\partial v}(\,\bar{r}_1.\bar{r}_2\,) - \tfrac{1}{2}\,\frac{\partial}{\partial u}(\,\bar{r}_2.\bar{r}_2\,)$$

$$\text{kurl } Q\bar{r}_2 = \frac{\bar{n}}{H}\,Q\bar{r}_2\bar{r}_{12} + \frac{\bar{n}}{H}\,Q\bar{r}_1\bar{r}_{22} + \frac{\bar{n}.\bar{r}_{22}}{H}\,\bar{r}_1 Q - \frac{\bar{n}.\bar{r}_{12}}{H}\,\bar{r}_2 Q$$

$$= \frac{\bar{n}Q}{H}\,\tfrac{1}{2}\,\frac{\partial}{\partial u}(\,\bar{r}_2.\bar{r}_2\,) - \frac{\bar{n}Q}{H}\,\frac{\partial}{\partial v}(\,\bar{r}_1.\bar{r}_2) -$$

$$- \frac{\bar{n}Q}{H}\,\tfrac{1}{2}\,\frac{\partial}{\partial u}(\,\bar{r}_2.\bar{r}_2) + \frac{Q}{H}(\,N\bar{r}_1 - M\bar{r}_2\,)$$

$$= \frac{\bar{n}Q}{H}\,\frac{\partial}{\partial u}(\,G\,) - \frac{\bar{n}Q}{H}\,\frac{\partial}{\partial v}(F) + \frac{Q}{H}(\,N\bar{r}_1 - M\bar{r}_2)$$

$$\text{kurl } Q\bar{r}_2 = \frac{1}{H}\left\{\frac{\partial}{\partial u}(GQ) - \frac{\partial}{\partial v}(FQ)\right\}\bar{n} +$$

$$\frac{Q}{H}(\,N\bar{r}_1 - M\bar{r}_2\,)$$

$$\text{kurl } \bar{\alpha} = \text{kurl } P\bar{r}_1 + \text{kurl } Q\bar{r}_2 + \text{kurl } R\bar{n}$$

$$= \frac{1}{H}\left\{\frac{\partial}{\partial u}(\,FP\,) - \frac{\partial}{\partial v}(\,EP\,)\right\}\bar{n} +$$

$$\frac{P}{H}(\,M\bar{r}_1 - L\bar{r}_2\,) + \frac{1}{H}\left\{\frac{\partial}{\partial u}(\,GQ\,) - \frac{\partial}{\partial v}(FQ)\right\}\bar{n} +$$

$$+ \frac{Q}{H} (N\bar{r}_1 - M\bar{r}_2) + \bar{\nabla} R \times \bar{n}$$

$$kurl\ \bar{\alpha} = \frac{1}{H} \left\{ \frac{\partial}{\partial u} (FP + GQ) - \frac{\partial}{\partial v} (EP + FQ) \right\} \bar{n} +$$

$$\frac{1}{H} (PM + QN) \bar{r}_1 - \frac{1}{H} (PL + QM) \bar{r}_2 +$$

$$(\bar{\nabla} R \times \bar{n}) \quad \ldots\ldots\ldots\ldots\ldots \quad (2.5.1)$$

bila $\bar{\alpha} = \bar{\nabla} \emptyset$

dan diketahui,

$$\bar{\nabla} \emptyset = \frac{(G\emptyset_1 - F\emptyset_2)}{H^2} \bar{r}_1 + \frac{(E\emptyset_2 - F\emptyset_1)}{H^2} \bar{r}_2$$

$$\ldots\ldots\ldots \quad (2.1.4)$$

$$kurl\ \bar{\alpha} = kurl\ \bar{\nabla} \emptyset$$

$$= \bar{\nabla} \times \bar{\nabla} \emptyset$$

dengan mensubtitusi nilai-nilai P dan Q dari persamaan (2.1.4) pada persamaan (2.5.1) didapat :

$$kurl\ \bar{\nabla} \emptyset = \frac{1}{H} \left\{ \frac{\partial}{\partial u} \left(\frac{G\emptyset_1 - F\emptyset_2}{H^2} \right) F + \frac{\partial}{\partial u} G \left(\frac{E\emptyset_2 - F\emptyset_1}{H^2} \right) \right\} -$$

$$\frac{1}{H} \left\{ \frac{\partial}{\partial v} E \left(\frac{G\emptyset_1 - F\emptyset_2}{H^2} \right) + \frac{\partial}{\partial v} F \left(\frac{E\emptyset_2 - F\emptyset_1}{H^2} \right) \right\}$$

$$= \frac{1}{H} \left\{ \frac{\partial}{\partial u} \left(\frac{GF\emptyset_1 - F^2\emptyset_2}{H^2} \right) + \frac{\partial}{\partial u} \left(\frac{EG\emptyset_2 - GF\emptyset_1}{H^2} \right) \right\} -$$

$$\frac{1}{H} \left\{ \frac{\partial}{\partial v} \left(\frac{EG\emptyset_1 - EF\emptyset_2}{H^2} \right) + \frac{\partial}{\partial v} \left(\frac{EF\emptyset_2 - F^2\emptyset_1}{H^2} \right) \right\}$$

$$= \frac{1}{H} \left\{ \frac{\partial}{\partial u} \left(\frac{EG - F^2}{H^2} \right) \emptyset_2 - \frac{\partial}{\partial v} \left(\frac{EG - F^2}{H^2} \right) \emptyset_1 \right\}$$

$$= \frac{1}{H} \frac{\partial}{\partial u} \emptyset_2 - \frac{1}{H} \frac{\partial}{\partial v} \emptyset_1$$

$$kurl\ \bar{\nabla} \emptyset = 0$$

Jadi kurl dari gradien fungsi skalar adalah tangensial pada permukaan.

II - 5. FUNGSI VEKTOR.

Operator $\bar{a}.\bar{\nabla}$ diterapkan pada suatu fungsi vektor, dimana diketahui bahwa $\bar{a}.\bar{\nabla}\phi$ adalah derivative ϕ pada arah $\bar{a}$.

$\bar{a}$ = unit vektor

$$\frac{d\phi}{ds} = \bar{a}.\bar{\nabla}\phi$$

Derivative $\bar{\alpha}$ pada arah $\bar{a}$,

$$\bar{a}.\nabla\bar{\alpha} = \frac{1}{H^2}\,\bar{a}.\bar{r}_1\left(G\frac{\partial\bar{\alpha}}{\partial u} - F\frac{\partial\bar{\alpha}}{\partial v}\right) + \frac{1}{H^2}\,\bar{a}.\bar{r}_2\left(E\frac{\partial\bar{\alpha}}{\partial v} - F\frac{\partial\bar{\alpha}}{\partial u}\right)$$

jika $\bar{a} = \bar{r}_1/\sqrt{E}$,

$$\bar{a}.\nabla\bar{\alpha} = \frac{1}{H^2\sqrt{E}}\left\{ E\,(G\bar{\alpha}_1 - F\bar{\alpha}_2) + F\,(E\bar{\alpha}_2 - F\bar{\alpha}_1)\right\}$$

$$= \frac{1}{H^2\sqrt{E}}\,(EG\bar{\alpha}_1 - EF\bar{\alpha}_2 + EF\bar{\alpha}_2 - F^2\bar{\alpha}_1)$$

$$= \frac{1}{H^2\sqrt{E}}\,(EG - F^2)\,\bar{\alpha}_1$$

$$\bar{a}.\nabla\bar{\alpha} = \frac{\bar{\alpha}_1}{\sqrt{E}} = \frac{1}{\sqrt{E}}\frac{\partial\bar{\alpha}}{\partial u}$$

Operator ∇^2 dalam persamaan (2.4.4) dapat diterapkan pula pada vektor fungsi titik yang memberikan suatu invarian diferensial order kedua.

Akan dihitung nilai dari $\nabla^2\bar{r}$.

Dari persamaan (2.4.5) diketahui :

$$\nabla^2\phi = \frac{1}{\sqrt{EG}}\frac{\partial}{\partial u}\left(\phi_1\sqrt{\frac{G}{E}}\right) + \frac{1}{\sqrt{EG}}\frac{\partial}{\partial v}\left(\phi_2\sqrt{\frac{E}{G}}\right)$$

$$\nabla^2\bar{r} = \frac{1}{\sqrt{EG}}\frac{\partial}{\partial u}\left(\bar{r}_1\sqrt{\frac{G}{E}}\right) + \frac{1}{\sqrt{EG}}\frac{\partial}{\partial v}\left(\bar{r}_2\sqrt{\frac{E}{G}}\right)$$

$$\bar{\nabla}^2\bar{r} = \frac{1}{\sqrt{EG}}\left[\left\{\frac{\partial}{\partial u}\left(\sqrt{\frac{G}{E}}\right)\bar{r}_1 + \frac{\partial}{\partial u}(\bar{r}_1)\sqrt{\frac{G}{E}}\right\} + \right.$$
$$\left. + \frac{\partial}{\partial v}\left(\sqrt{\frac{E}{G}}\right)\bar{r}_2 + \frac{\partial}{\partial v}(\bar{r}_2)\sqrt{\frac{E}{G}}\right]$$

$$\bar{\nabla}^2\bar{r} = \frac{1}{\sqrt{EG}}\left\{\bar{r}_1\left(\frac{G_1}{2\sqrt{EG}} - \frac{E_1\sqrt{G}}{2E\sqrt{E}}\right) + \sqrt{\frac{G}{E}}\,\bar{r}_{11} + \bar{r}_2\left(\frac{E_2}{2\sqrt{EG}} - \right.\right.$$
$$\left.\left. - \frac{G_2\sqrt{E}}{2G\sqrt{G}}\right) + \sqrt{\frac{E}{G}}\,\bar{r}_{22}\right\}$$

$$\bar{\nabla}^2\bar{r} = \frac{1}{\sqrt{EG}}\left\{\sqrt{\frac{E}{G}}\left(\frac{G_1}{2E}\bar{r}_1 - \frac{G_2}{2G}\bar{r}_2\right) + \sqrt{\frac{G}{E}}\left(\frac{E_2}{2G}\bar{r}_2 - \frac{E_1}{2E}\bar{r}_1\right) + \right.$$
$$\left. \sqrt{\frac{G}{E}}\,\bar{r}_{11} + \sqrt{\frac{E}{G}}\,\bar{r}_{22}\right\}$$

dengan formula Gauss didapatkan harga $\bar{r}_{11}$, $\bar{r}_{22}$

$\bar{r}_{11} = L\bar{n} + l\bar{r}_1 + \lambda\bar{r}_2$

$\bar{r}_{12} = M\bar{n} + m\bar{r}_1 + \mu\,\bar{r}_2$

$\bar{r}_{22} = N\bar{n} + k\bar{r}_1 + \jmath\,\bar{r}_2$

$\bar{r}_1 \cdot \bar{r}_{11} = \frac{1}{2}E_1$

$\bar{r}_2 \cdot \bar{r}_{11} = F_1 - \frac{1}{2}E_2$

$\frac{1}{2}E_1 = l.\bar{r}_1.\bar{r}_1 + \lambda\,\bar{r}_2.\bar{r}_1 = lE + \lambda F$

$F_1 - \frac{1}{2}E_2 = l.\bar{r}_1.\bar{r}_2 + \lambda\bar{r}_2.\bar{r}_2 = lF + \lambda G$

$\frac{1}{2}E_1G = lEG + \lambda GF$ ===> 1 x G

$FF_1 - \frac{1}{2}E_2F = lF^2 + \lambda GF$ - ===> 2 x F

$\frac{1}{2}E_1G - FF_1 + \frac{1}{2}E_2F = lH^2$

$$l = \frac{1}{2H^2}(GE_1 - 2FF_1 + FE_2)$$

$EF_1 - \frac{1}{2}EE_2 = l\ EF + \lambda\ EG$ ===> 2 x E

$\frac{1}{2}E_1F = l\ EF + \lambda F^2$ - ===> 1 x F

$EF_1 - \frac{1}{2}EE_2 - \frac{1}{2}E_1F = \lambda H^2$

$$\lambda = \frac{1}{2H^2} \left(2EF_1 - EE_2 - FE_1 \right)$$

$$\bar{r}_1 \cdot \bar{r}_{22} = F_2 - \tfrac{1}{2}G_1$$

$$\bar{r}_2 \cdot \bar{r}_{22} = \tfrac{1}{2}G_2$$

$$F_2 - \tfrac{1}{2}G_1 = k.\bar{r}_1 .\bar{r}_1 + j\bar{r}_1 .\bar{r}_2 = kE + jF$$

$$\tfrac{1}{2}G_2 = k.\bar{r}_1 .\bar{r}_2 + j\,\bar{r}_2 .\bar{r}_2 = kF + jG$$

$$2 \times E \Longrightarrow \tfrac{1}{2}G_2 E = k\,EF + jEG$$

$$1 \times F \Longrightarrow FF_2 - \tfrac{1}{2}FG_1 = k\,EF + jF^2 \quad -$$

$$\tfrac{1}{2}EG_2 - FF_2 + \tfrac{1}{2}FG_1 = jH^2$$

$$j = \frac{1}{2H^2} \left(EG_2 - 2FF_2 + FG_1 \right)$$

$$1 \times G \Longrightarrow GF_2 - \tfrac{1}{2}GG_1 = k.EG + jFG$$

$$2 \times F \Longrightarrow \tfrac{1}{2}FG_2 = k.F^2 + jFG \quad -$$

$$GF_2 - \tfrac{1}{2}GG_1 - \tfrac{1}{2}FG_2 = kH^2$$

$$k = \frac{1}{2H^2} \left(2GF_2 - GG_1 - FG_2 \right)$$

karena kurva parametrik orthogonal,

$$F = 0$$

$$H^2 = EG$$

$$\bar{r}_{11} = L\bar{n} + \frac{E_1}{2E}\bar{r}_1 - \frac{E_2}{2G}\bar{r}_2$$

$$\bar{r}_{22} = N\bar{n} - \frac{G_1}{2E}\bar{r}_1 + \frac{G_2}{2G}\bar{r}_2$$

$$\bar{\nabla}^2 \bar{r} = \frac{1}{\sqrt{EG}} \left\{ \sqrt{\frac{E}{G}} \left(\frac{G_1}{2E}\bar{r}_1 - \frac{G_2}{2G}\bar{r}_2 \right) + \sqrt{\frac{G}{E}} \left(\frac{E_2}{2G}\bar{r}_2 - \frac{E_1}{2E}\bar{r}_1 \right) \right.$$

$$+ \sqrt{\frac{G}{E}} \left(L\bar{n} + \frac{E_1}{2E}\bar{r}_1 - \frac{E_2}{2G}\bar{r}_2 \right) + \sqrt{\frac{E}{G}} \left(N\bar{n} - \right.$$

$$\left. \frac{G_1}{2E}\bar{r}_1 + \frac{G_2}{2G}\bar{r}_2 \right) \right\}$$

$$\bar{\nabla}^2 \bar{r} = \frac{L}{E}\bar{n} + \frac{N}{G}\bar{n} = \left(\frac{L}{E} + \frac{N}{G} \right)\bar{n}$$

$$\nabla^2 \bar{r} = \gamma \bar{n} \qquad \ldots\ldots \quad (2.6.1)$$

diketahui (2.4.2),

$$\text{div } R\bar{n} = R.\text{div } \bar{n} = -\gamma R$$

$$\nabla.\nabla^2\bar{r} = \nabla.\gamma\bar{n}$$

$$= \gamma\nabla.\bar{n}$$

$$\nabla.\nabla^2\bar{r} = -\gamma^2$$

II-6. RUMUS - RUMUS DIFERENSIASI.

Bila $\emptyset$ sebuah fungsi skalar dan $\bar{A}$, $\bar{B}$ fungsi - fungsi vektor, dengan penguraian langsung, persa - maan -persamaan dibawah ini dapat dibuktikan :

$$\nabla.(\emptyset\bar{A}) = \nabla\emptyset.\bar{A} + \emptyset\nabla.\bar{A} \qquad \ldots\ldots\ldots \quad (1)$$

$$\nabla\times(\emptyset\bar{A}) = \nabla\emptyset\times\bar{A} + \emptyset\nabla\times\bar{A} \qquad \ldots\ldots\ldots \quad (2)$$

$$\nabla.(\bar{A}\times\bar{B}) = \bar{B}.\nabla\times\bar{A} - \bar{A}.\nabla\times\bar{B} \qquad \ldots \quad (3)$$

Akan dibuktikan persamaan-persamaan diatas dengan opera- tor diferensial vektorial dan dengan mengambil kurva

parametrik orthogonal,

$F = 0$

$H^2 = EG$

$$\nabla = \frac{1}{E}\bar{r}_1\frac{\partial}{\partial u} + \frac{1}{G}\bar{r}_2\frac{\partial}{\partial v}$$

bukti :

1. $$\nabla.(\emptyset\bar{A}) = \frac{1}{E}\bar{r}_1\frac{\partial}{\partial u}(\emptyset\bar{A}) + \frac{1}{G}\bar{r}_2\frac{\partial}{\partial v}(\emptyset\bar{A})$$

$$= \frac{1}{E}\bar{r}_1(\emptyset_1\bar{A} + \emptyset\bar{A}_1) + \frac{1}{G}\bar{r}_2(\emptyset_2\bar{A} + \emptyset\bar{A}_2)$$

$$= \left(\frac{1}{E}\bar{r}_1\emptyset_1 + \frac{1}{G}\bar{r}_2\emptyset_2\right).\bar{A} + \emptyset\left(\frac{1}{E}\bar{r}_1\bar{A}_1 + \frac{1}{G}\bar{r}_2.\bar{A}_2\right)$$

$$\nabla.(\emptyset\bar{A}) = \nabla\emptyset.\bar{A} + \emptyset\nabla.\bar{A}$$

2. $$\bar{\nabla} \times (\emptyset \bar{A}) = \frac{1}{E}\bar{r}_1 \times \frac{\partial}{\partial u}(\emptyset \bar{A}) + \frac{1}{G}\bar{r}_2 \times \frac{\partial}{\partial v}(\emptyset \bar{A})$$

$$= \frac{1}{E}\bar{r}_1 \times (\emptyset_1 \bar{A} + \emptyset \bar{A}_1) + \frac{1}{G}\bar{r}_2 \times (\emptyset_2 \bar{A} + \emptyset \bar{A}_2)$$

$$= \left(\frac{1}{E}\bar{r}_1 \cdot \emptyset_1 + \frac{1}{G}\bar{r}_2 . \emptyset_2\right) \times \bar{A} + \left(\frac{1}{E}\bar{r}_1 \times \bar{A}_1 + \frac{1}{G}\bar{r}_2 \times \bar{A}_2\right)\emptyset$$

$$\bar{\nabla} \times (\emptyset \bar{A}) = \bar{\nabla}\emptyset \times \bar{A} + \emptyset(\bar{\nabla} \times \bar{A})$$

3. $$\bar{\nabla}(\bar{A} \times \bar{B}) = \frac{1}{E}\bar{r}_1 . \frac{\partial}{\partial u}(\bar{A} \times \bar{B}) + \frac{1}{G}\bar{r}_2 . \frac{\partial}{\partial v}(\bar{A} \times \bar{B})$$

$$= \frac{1}{E}\bar{r}_1 (\bar{A}_1 \times \bar{B} + \bar{A} \times \bar{B}_1) + \frac{1}{G}\bar{r}_2 (\bar{A}_2 \times \bar{B} + \bar{A} \times \bar{B}_2)$$

$$\bar{\nabla} . (\bar{A} \times \bar{B}) = \left(\frac{1}{E}\bar{r}_1 \times \bar{A}_1 + \frac{1}{G}\bar{r}_2 \times \bar{A}_2\right)\bar{B} - \left(\frac{1}{E}\bar{r}_1 \times \bar{B}_1 + \frac{1}{G}\bar{r}_2 \times \bar{B}_2\right)\bar{A}$$

$$\bar{\nabla} . (\bar{A} \times \bar{B}) = \bar{B} . \bar{\nabla} \times \bar{A} - \bar{A} . \bar{\nabla} \times \bar{B}$$

Sebagai contoh diterapkan (1) dan (2) pada fungsi $R\bar{n}$

- $\text{div } R\bar{n} = \bar{\nabla} R . \bar{n} = \bar{\nabla} R . \bar{n} + R\, \bar{\nabla} . \bar{n}$

  $\bar{\nabla} R \perp \bar{n}$, $\text{div } \bar{n} = -J$

  $\text{div } R\bar{n} = -J R$

- $\text{kurl } R\bar{n} = \bar{\nabla} \times R\bar{n}$

  $= \bar{\nabla} R \times \bar{n} + R\, \bar{\nabla} \times \bar{n}$

  $\text{kurl } \bar{n} = 0$

  $\text{kurl } R\bar{n} = \bar{\nabla} R \times \bar{n}$

- $\text{div kurl } R\bar{n} = \text{div } (\bar{\nabla} R \times \bar{n})$

$$\text{div kurl } R\bar{n} = \bar{\nabla}.(\bar{\nabla}R \times \bar{n})$$

$$= \bar{n}.\bar{\nabla} \times \bar{\nabla}R - \bar{\nabla}R.\bar{\nabla} \times \bar{n}$$

$$\text{div kurl } R\bar{n} = 0 \quad \ldots\ldots\ldots\ldots (2.7.1)$$

Contoh :

1. Perlihatkan bahwa : div $\bar{r} = 2$

- kurva parametrik orthogonal,

$$\text{div } \bar{r} = \frac{1}{E}\bar{r}_1 \frac{\partial}{\partial u}(\bar{r}) + \frac{1}{G}\bar{r}_2 \frac{\partial}{\partial v}(\bar{r})$$

$$= \frac{1}{E}\bar{r}_1 . \bar{r}_1 + \frac{1}{G}\bar{r}_2 . \bar{r}_2$$

$$\text{div } \bar{r} = \frac{E}{E} + \frac{G}{G} = 2$$

2. Perlihatkan : div $\partial\bar{r} = \bar{r}.\bar{\nabla}\partial + 2\partial$

kurl $\partial\bar{r} = \bar{\nabla}\partial \times \bar{r}$

Penyelesaian :

$$\text{div } \partial\bar{r} = \bar{\nabla}.(\partial\bar{r})$$

$$= \bar{\nabla}\partial.\bar{r} + \partial.\bar{\nabla}\bar{r}$$

$$\text{div } \partial\bar{r} = \bar{r}.\bar{\nabla}\partial + 2\partial$$

$$\text{kurl } \partial\bar{r} = \bar{\nabla} \times \partial\bar{r}$$

$$= \bar{\nabla}\partial \times \bar{r} + \partial.\bar{\nabla} \times \bar{r}$$

$$= \bar{\nabla}\partial \times \bar{r} + 0$$

$$\text{kurl } \partial\bar{r} = \bar{\nabla}\partial \times \bar{r}$$

3. Perlihatkan : $\bar{\nabla}(\phi\psi) = \phi\bar{\nabla}\psi + \psi\bar{\nabla}\phi$

Penyelesaian :

$$\bar{\nabla}(\phi\psi) = \frac{1}{E}\bar{r}_1(\phi_1\psi + \phi\psi_1) + \frac{1}{G}\bar{r}_2 (\phi_2\psi + \phi\psi_2)$$

$$= \left(\frac{1}{E}\bar{r}_1\phi_1 + \frac{1}{G}\bar{r}_2\phi_2\right)\psi +$$

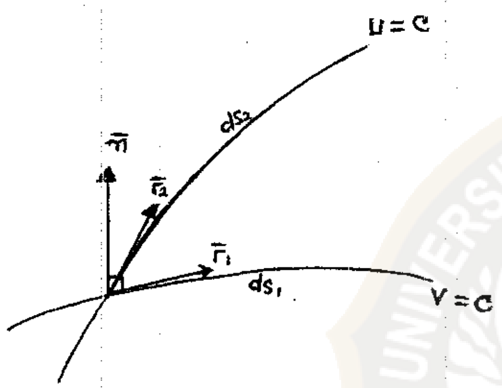
$$\phi\left(\frac{1}{E}\bar{r}_1\psi_1 + \frac{1}{G}\bar{r}_2\psi_2\right)$$

$$\bar{\nabla}(\phi\psi) = \bar{\nabla}\phi\,\psi + \phi\bar{\nabla}\psi$$

$$\bar{\nabla}(\phi\psi) = \phi\bar{\nabla}\psi + \psi\bar{\nabla}\phi$$

II - 7. KELENGKUNGAN GEODETIK .

Ditentukan sistim orthogonal dari kurva parameter, $\bar{a},\bar{b}$ adalah unit vektor pada arah $\bar{r}_1$ dan $\bar{r}_2$



$\bar{r}_1 = \bar{r}_u$, $\bar{r}_2 = \bar{r}_v$

elemen garis :

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2$$

elemen garis pada v=c

$$ds^2 = Edu^2$$

$$ds_1 = \sqrt{E}\,du$$

$$\frac{ds_1}{du} = |\bar{r}_1| = \sqrt{E}$$

maka: $\bar{a} = \dfrac{\bar{r}_1}{\sqrt{E}}$

elemen garis pada u=c

$$ds_2 = \sqrt{G}\,dv$$

$$\frac{ds_2}{dv} = |\bar{r}_2| = \sqrt{G}$$

$$\bar{b} = \frac{\bar{r}_2}{\sqrt{G}}$$

$\bar{a},\bar{b},\bar{n}$ membentuk sistim putar kanan :

$$\bar{a} x \bar{b} = \bar{n}$$

$$\bar{b} x \bar{n} = \bar{a}$$

$$\bar{n} x \bar{a} = \bar{b}$$

Vektor kelengkungan dari kurva parameter v = konstan,

$$\bar{K} = \frac{1}{\sqrt{E}} \frac{\partial \bar{a}}{\partial u}$$

$$\frac{\partial \bar{a}}{\partial u} = \frac{\partial}{\partial u}\left(\frac{\bar{r}_1}{\sqrt{E}}\right) = \frac{1}{\sqrt{E}}\bar{r}_{11} - \frac{E_1}{2E\sqrt{E}}\bar{r}_1$$

$\bar{r}_{11}$ dihitung terlebih dahulu :

$$\bar{r}_{11} = L\bar{n} + l\bar{r}_1 + \lambda\bar{r}_2$$

$$\bar{r}_{12} = M\bar{n} + m\bar{r}_1 + \mu\bar{r}_2$$

diketahui bahwa :

$$\bar{r}_1 \cdot \bar{r}_{11} = \tfrac{1}{2} E_1$$

$$\bar{r}_2 \cdot \bar{r}_{11} = F_1 - \tfrac{1}{2} E_2$$

$$\bar{r}_1 \cdot \bar{r}_{11} = (L\bar{n} + l\bar{r}_1 + \lambda\bar{r}_2) \bar{r}_1$$

$$\bar{r}_1 \cdot \bar{r}_{11} = lE + \lambda F = \tfrac{1}{2} E_1$$

$$\bar{r}_2 \cdot \bar{r}_{11} = (L\bar{n} + l\bar{r}_1 + \lambda\bar{r}_2) \bar{r}_2$$

$$\bar{r}_2 \cdot \bar{r}_{11} = lF + \lambda G = F_1 - \tfrac{1}{2}E_2$$

$$1 \times F \Longrightarrow lEF + \lambda F^2 = \tfrac{1}{2}E_1 F$$

$$2 \times E \Longrightarrow \underline{lEF + \lambda EG = EF_1 - \tfrac{1}{2}EE_2} \; -$$

$$\lambda (EG - F^2) = EF_1 - \tfrac{1}{2} EE_2 - \tfrac{1}{2} E_1 F$$

$$\lambda = \frac{1}{2H^2} (2EF_1 - EE_2 - FE_1)$$

$$1 \times G \Longrightarrow lEG + \lambda FG = \tfrac{1}{2}E_1 G$$

$$2 \times F \Longrightarrow \underline{lF^2 + \lambda FG = FF_1 - \tfrac{1}{2}E_2 F} \; -$$

$$l (EG - F^2) = \tfrac{1}{2}E_1 G - FF_1 + \tfrac{1}{2}FE_2$$

$$l = \frac{1}{2H^2} (GE_1 - 2FF_1 + FE_2)$$

juga diketahui bahwa :

$$\bar{r}_1 \cdot \bar{r}_{12} = \tfrac{1}{2}E_2$$

$$\bar{r}_2 \cdot \bar{r}_{12} = \tfrac{1}{2}G_1$$

$$\bar{r}_1 \cdot \bar{r}_{12} = (M\bar{n} + m\bar{r}_1 + \mu\bar{r}_2)\bar{r}_1$$

$$= mE + \mu F = \tfrac{1}{2}E_2$$

$$\bar{r}_2 \cdot \bar{r}_{12} = (M\bar{n} + m\bar{r}_1 + \mu\bar{r}_2)\bar{r}_2$$

$$= mF + \mu G = \tfrac{1}{2}G_1$$

$$3 \times F \Longrightarrow mEF + \mu F^2 = \tfrac{1}{2}E_2 F$$

$$4 \times E \Longrightarrow \underline{mEF + \mu EG = \tfrac{1}{2}EG_1} \; -$$

$$\mu (EG - F^2) = \tfrac{1}{2} (EG_1 - E_2 F)$$

$$\mu = \frac{1}{2H^2} (EG_1 - FE_2)$$

$$3 \times G \Longrightarrow \quad mEG + \mu FG = \tfrac{1}{2}E_2G$$

$$4 \times F \Longrightarrow \quad mF^2 + \mu FG = \tfrac{1}{2}FG_1 \quad -$$

$$m (EG - F^2) = \tfrac{1}{2} (GE_2 - FG_1)$$

$$m = \frac{1}{2H^2} (GE_2 - FG_1)$$

karena kurva parameter orthogonal :

$$F = 0$$

$$H^2 = EG \quad \text{maka :}$$

$$\bar{r}_{11} = L\bar{n} + \frac{E_1}{2E}\bar{r}_1 - \frac{E_2}{2G}\bar{r}_2$$

$$\bar{r}_{12} = M\bar{n} + \frac{E_2}{2E}\bar{r}_1 + \frac{G_1}{2G}\bar{r}_2$$

$$\frac{\partial \bar{a}}{\partial u} = \frac{1}{\sqrt{E}} (L\bar{n} + \frac{E_1}{2E}\bar{r}_1 - \frac{E_2}{2G}\bar{r}_2) - \frac{E_1\bar{r}_1}{2E\sqrt{E}}$$

$$= \frac{L\bar{n}}{\sqrt{E}} + \frac{E_1}{2E\sqrt{E}}\bar{r}_1 - \frac{E_2}{2G\sqrt{E}}\bar{r}_2 - \frac{E_1\bar{r}_1}{2E\sqrt{E}}$$

$$\frac{\partial \bar{a}}{\partial u} = \frac{L}{\sqrt{E}}\bar{n} - \frac{E_2}{2G\sqrt{E}}\bar{r}_2$$

$$\bar{b} = \frac{\bar{r}_2}{\sqrt{G}}$$

$$\bar{r}_2 = \bar{b}\sqrt{G}$$

$$\frac{\partial \bar{a}}{\partial u} = \frac{L}{\sqrt{E}}\bar{n} - \frac{E_2}{2G\sqrt{E}}\bar{b}\sqrt{G}$$

$$= \frac{L}{\sqrt{E}}\bar{n} - \frac{E_2}{2\sqrt{EG}}\bar{b}$$

jadi vektor kelengkungan :

$$\bar{K} = \frac{L}{E}\,\bar{n} - \frac{E_2}{2E\sqrt{G}}\,\bar{b}$$

$$\bar{K}n = \frac{L}{E}$$

$$\bar{K}g = \frac{-E_2}{2E\sqrt{G}}$$

Kelengkungan geodetik dari kurva parameter v = konstan adalah negatif dari divergen unit vektor $\bar{b}$.

Dari (2.4.3) diketahui :

$$\text{div}\ \bar{\alpha} = \frac{1}{H}\left\{ \frac{\partial}{\partial u}(HP) + \frac{\partial}{\partial v}(HQ)\right\} - \gamma R$$

$$\bar{b} = \frac{\bar{r}_2}{\sqrt{G}}$$

$$\text{div}\ \bar{b} = \text{div}\left(\frac{\bar{r}_2}{\sqrt{G}} \right)$$

$$= \frac{1}{H}\frac{\partial}{\partial v}\left(\frac{H}{\sqrt{G}} \right)$$

karena F = 0, maka :

$$\text{div}\left(\frac{\bar{r}_2}{\sqrt{G}} \right) = \frac{1}{\sqrt{EG}}\frac{\partial}{\partial v}\frac{\sqrt{EG}}{\sqrt{G}} = \frac{1}{\sqrt{EG}}\frac{\partial}{\partial v}\sqrt{E}$$

$$= \frac{1}{\sqrt{EG}}\, E_2 .\ \tfrac{1}{2}\, E^{-\frac{1}{2}}$$

$$= \frac{E_2}{2E\sqrt{G}}$$

$$\text{div}\left(\frac{\bar{r}_2}{\sqrt{G}} \right) = -\bar{K}g$$

Hasil yang sama dapat diperoleh dengan menganggap u = konstan

Vektor kelengkungan :

$$\bar{K} = \frac{1}{\sqrt{G}} \frac{\partial \bar{b}}{\partial v}$$

$$\frac{\partial \bar{b}}{\partial v} = \frac{\partial}{\partial v} \left(\frac{\bar{r}_2}{\sqrt{G}} \right) = \frac{1}{\sqrt{G}} \bar{r}_{22} - \frac{G_2 \bar{r}_2}{2G\sqrt{G}}$$

$\bar{r}_{22}$ dihitung terlebih dulu :

$\bar{r}_{22} = N\bar{n} + p\bar{r}_1 + q\bar{r}_2$

$\bar{r}_1 \cdot \bar{r}_{22} = F_2 - \frac{1}{2}G_1$

$\bar{r}_2 \cdot \bar{r}_{22} = \frac{1}{2}G_2$

$\bar{r}_1 \cdot \bar{r}_{22} = (N\bar{n} + p\bar{r}_1 + q\bar{r}_2)\bar{r}_1$

$\bar{r}_1 \cdot \bar{r}_{22} = pE + qF$

$\quad = F_2 - \frac{1}{2}G_1$

$\bar{r}_2 \cdot \bar{r}_{22} = pF + qG = \frac{1}{2}G_2$

$1 \times F \implies \quad pEF + qF^2 = FF_2 - \frac{1}{2}FG_1$

$2 \times E \implies \quad pEF + qEG = \frac{1}{2}EG_2 \quad -$

$q (F^2 - EG) = FF_2 - \frac{1}{2}FG_1 - \frac{1}{2}EG_2$

$q (EG - F^2) = \frac{1}{2}FG_1 + \frac{1}{2}EG_2 - FF_2$

$$q = \frac{1}{2H^2} (FG_1 + EG_2 - 2FF_2)$$

$1 \times G \implies \quad pEG + qFG = F_2 G - \frac{1}{2}GG_1$

$2 \times F \implies \quad pF^2 + qFG = \frac{1}{2}FG_2 \quad -$

$p (EG - F^2) = F_2 G - \frac{1}{2}GG_1 - \frac{1}{2}FG_2$

$$p = \frac{1}{2H^2} (2F_2 G - GG_1 - FG_2)$$

kurva parameter orthogonal :

$F = 0$

$H^2 = EG$

$$\bar{r}_{22} = N\bar{n} + \frac{1}{2EG} (- GG_1)\bar{r}_1 + \frac{1}{2EG} (EG_2)\bar{r}_2$$

$$\bar{r}_{22} = N\bar{n} - \frac{G_1}{2E}\bar{r}_1 + \frac{G_2}{2G}\bar{r}_2$$

$$\frac{\partial \bar{b}}{\partial v} = \frac{1}{\sqrt{G}}\left(N\bar{n} - \frac{G_1}{2E}\bar{r}_1 + \frac{G_2}{2G}\bar{r}_2 \right) - \frac{G_2 . \bar{r}_2}{2G\sqrt{G}}$$

$$= \frac{N\bar{n}}{\sqrt{G}} - \frac{G_1}{2E\sqrt{G}}\bar{r}_1 + \frac{G_2}{2G\sqrt{G}}\bar{r}_2 - \frac{G_2 . \bar{r}_2}{2G\sqrt{G}}$$

$$\frac{\partial \bar{b}}{\partial v} = \frac{N\bar{n}}{\sqrt{G}} - \frac{G_1}{2E\sqrt{G}}\bar{r}_1$$

diketahui :

$$\bar{a} = \frac{\bar{r}_1}{\sqrt{E}} \Longrightarrow \bar{r}_1 = \bar{a}\sqrt{E}$$

$$\bar{K} = \frac{1}{\sqrt{G}}\left(\frac{N\bar{n}}{\sqrt{G}} - \frac{G_1}{2E\sqrt{G}} . \bar{a}\sqrt{E} \right)$$

$$\bar{K} = \frac{N}{G}\bar{n} - \frac{G_1}{2G\sqrt{E}}\bar{a}$$

kelengkungan geodetik dari u = konstan:

$$\bar{Kg} = \frac{G_1}{2G\sqrt{E}}$$

$$\bar{Kg} = \text{div } \bar{a}$$

$$= - \text{div } (-\bar{a})$$

$$\text{div } \bar{a} = \text{div } \left(\frac{\bar{r}_1}{\sqrt{E}} \right)$$

$$= \frac{1}{H}\frac{\partial}{\partial u}\left(\frac{H}{\sqrt{E}} \right)$$

F = 0

$$\text{div } \bar{a} = \frac{1}{\sqrt{EG}}\frac{\partial}{\partial u}\left(\frac{\sqrt{EG}}{\sqrt{E}} \right)$$

$$= \frac{1}{\sqrt{EG}}\frac{\partial}{\partial u}\sqrt{G}$$

$$= \frac{1}{\sqrt{EG}} G_1 . \tfrac{1}{2}G^{-\frac{1}{2}}$$

$$\text{div } \bar{a} = \frac{G_1}{2G\sqrt{E}}$$

Jadi kelengkungan geodetik dari kurva parameter u = konstan adalah sama dengan div $\bar{a}$.

Pandang Ø (u,v) = konstan.

Kelengkungan geodetik dari Ø (u,v) didapat dengan menggunakan rumus Bonnet's .

Vektor satuan tegak lurus pada kurva adalah : $\dfrac{\bar{\nabla} \phi}{|\bar{\nabla}\phi|}$

$$|\bar{\nabla} \phi| = \frac{1}{H}\sqrt{E\phi_2^2 - 2F\phi_1\phi_2 + G\phi_1^2}$$

$$\bar{\nabla}^2 \phi = \frac{1}{H}\frac{\partial}{\partial u}\left(\frac{G\phi_1 - F\phi_2}{H}\right) + \frac{1}{H}\frac{\partial}{\partial v}\left(\frac{E\phi_2 - F\phi_1}{H}\right)$$

$$\bar{K}g = \text{div } \frac{\bar{\nabla}\phi}{|\bar{\nabla}\phi|} = \bar{\nabla} \cdot \frac{\bar{\nabla}\phi}{|\bar{\nabla}\phi|}$$

$$\bar{K}g = \frac{\frac{1}{H}\frac{\partial}{\partial u}\left(\frac{G\phi_1 - F\phi_2}{H}\right) + \frac{1}{H}\frac{\partial}{\partial v}\left(\frac{E\phi_2 - F\phi_1}{H}\right)}{\frac{1}{H}\sqrt{E\phi_2^2 - 2F\phi_1\phi_2 + G\phi_1^2}}$$

$$\bar{K}g = \frac{1}{H}\left\{\frac{\partial}{\partial u}\left(\frac{G\phi_1 - F\phi_2}{\sqrt{E\phi_2^2 - 2F\phi_1\phi_2 + G\phi_1^2}}\right) + \frac{\partial}{\partial v}\left(\frac{E\phi_2 - F\phi_1}{\sqrt{E\phi_2^2 - 2F\phi_1\phi_2 + G\phi_1^2}}\right)\right\}$$

Bila $\bar{t}$ adalah garis singgung satuan, vektor satuan tegak lurus pada kurva adalah $\bar{n} \times \bar{t}$.

$$\bar{K}g = - \text{div } (\bar{n} \times \bar{t})$$

$$= - \bar{\nabla} \cdot (\bar{n} \times \bar{t})$$

$$= \bar{n} \cdot \bar{\nabla} \times \bar{t} - \bar{t} \cdot \bar{\nabla} \times \bar{n}$$

$$\text{kurl } \bar{n} = \bar{\nabla} \times \bar{n} = 0$$

maka :

$$\overline{Kg} = \bar{n} \cdot \nabla \times \bar{t}$$
$$= \bar{n} \cdot \text{kurl } \bar{t}$$

Maka didapatkan suatu teorema :

Bahwa jika diberikan famili kurva pada permukaan dengan tanda positif sepanjang kurva, kelengkungan geodetik dari anggota famili adalah hasil kali skalar dari normal dengan kurl garis singgung.