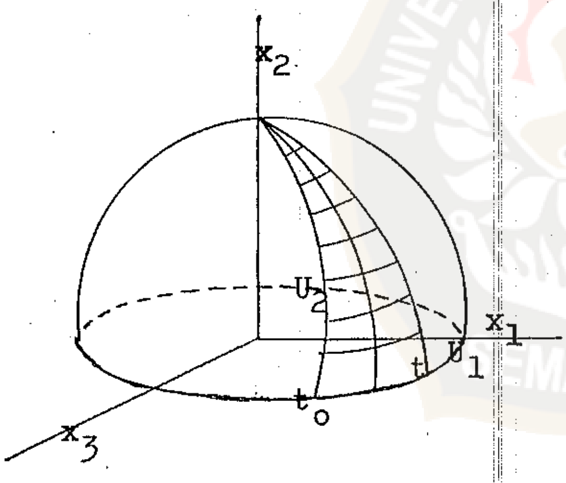
BAB II

II.1. ELEMEN LINIER DARI SUATU PERMUKAAN.

Pandang persamaan $x^i = f^i(u^1, u^2)$ .... (2.1)

dimana x^i menyatakan koordinat Cartesius dan $i = 1,2,3$. Bila persamaan (2.1) adalah suatu permukaan dengan batas batas kurva u^1 dan u^2 sebagai fungsi dari t, maka persamaan kurva tersebut adalah :

$$u^\alpha = Q^\alpha(t)\ ,\ \alpha = 1,2 \quad \text{.....................} \quad (2.2)$$



Panjang elemen ds dari suatu kurva dalam ruang adalah :

$$ds^2 = \sum_i dx^i \ dX^i$$

Dari (2.1) kita peroleh :

$$dx^i = \frac{\partial f^i}{\partial u^\alpha} du^\alpha =$$

$$= \frac{\partial x^i}{\partial u^\alpha} du^\alpha \quad \text{......} \quad (2.3)$$

Sehingga :

$$ds^2 = g_{\alpha\beta}\, du^\alpha\, du^\beta \quad \text{.........................} \quad (2.4)$$

Persamaan ini disebut <u>Elemen Linier Dari Suatu Permukaan</u>.

Dimana :

$$g_{\alpha\beta} = \sum_i \frac{\partial x^i}{\partial u^\alpha} \frac{\partial X^i}{\partial u^\beta} \quad \text{...................} \quad (2.5)$$

$g_{\alpha\beta}\, du^\alpha\, du^\beta$ disebut <u>Bentuk Dasar Kwadratik I</u>.

Jika persamaan (2.2) kita diferensialkan ke t, maka :

$$\frac{du^\alpha}{dt} = \frac{dQ^\alpha(t)}{dt} \text{ atau } du^\alpha = \frac{dQ^\alpha(t)}{dt} dt \quad \text{.....} \quad (2.6)$$

Persamaan (2.6) disubstittusikan ke persamaan :

$$ds^2 = \sum_i \frac{\partial x^i}{\partial u^\alpha} du^\alpha \frac{\partial x^i}{\partial u^\beta} du^\beta$$

$$= \frac{dQ^\alpha(t)}{dt} \frac{dQ^\beta(t)}{dt} dt^2 \sum_i \frac{\partial x^i}{\partial u^\alpha} \frac{\partial x^i}{\partial u^\beta}$$

$$= g_{\alpha\beta} Q^{\alpha\prime} Q^{\beta\prime} dt^2$$

$$ds = \sqrt{g_{\alpha\beta} Q^{\alpha\prime} Q^{\beta\prime}} dt$$

Elemen linier dari suatu permukaan selanjutnya merupakan suatu busur dari suatu permukaan, yaitu :

$$s = \int_{t_0}^{t} \sqrt{g_{\alpha\beta} Q^{\alpha\prime} Q^{\beta\prime}} dt \quad \ldots\ldots\ldots\ldots\ldots (2.7)$$

Kembali kita perhatikan persamaan (2.4) ;

Bila u^2 = konstan, maka $du^2 = 0$

$$ds_1 = g_{11} du^1 du^1 \quad \text{atau} \quad ds_1 = g_{11} (du^1)^2$$

Dengan cara yang sama yaitu u^1 = konstan, diperoleh pula

$$ds_2 = g_{22} du^2 du^2 \quad \text{atau} \quad ds_2 = g_{22} (du^2)^2$$

Dari :

$$\left.\begin{aligned} ds_1 &= g_{11} (du^1)^2 \\ ds_2 &= g_{22} (du^2)^2 \end{aligned}\right\} \text{harus dipenuhi } g_{11} > 0 \text{ dan } g_{22} > 0$$

Jika tidak dipenuhi, kurva - kurva koordinat disebut <u>KURVA-KURVA MINIMAL</u>.

Perhatikan koordinat baru u'^1, u'^2 dan didefinisikan persamaan :

$$u^\alpha = \psi(u'^1, u'^2);\ \alpha = 1,2 \quad \ldots\ldots\ldots\ldots\ldots (2.8)$$

asalkan Jacobiannya, yaitu :

$$\frac{\partial(\psi^1, \psi^2)}{\partial(u'^1, u'^2)} \neq 0$$

$$du^{\alpha} = \frac{\partial u^{\alpha}}{\partial u'^{\gamma}} du'^{\gamma} \; ; \; \alpha, \gamma = 1,2 \quad \ldots\ldots\ldots\ldots \quad (2.9)$$

Disubstitusikan ke persamaan :

$$ds^2 = g_{\alpha\beta}\, du^{\alpha}\, du^{\beta} = g_{\alpha\beta} \frac{\partial u^{\alpha}}{\partial u'^{\gamma}} du'^{\gamma} \; \frac{\partial u^{\beta}}{\partial u'^{\delta}} du'^{\delta}$$

$$ds^2 = g_{\alpha\beta} \frac{\partial u^{\alpha}}{\partial u'^{\gamma}} \frac{\partial u^{\beta}}{\partial u'^{\delta}} du'^{\gamma}\, du'^{\delta} \quad \ldots\ldots\ldots\ldots (2.10)$$

$$ds^2 = g'_{\gamma\delta}\, du'^{\gamma}\, du'^{\delta} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.10a)$$

dimana $g'_{\gamma\delta} = g_{\alpha\beta} \dfrac{\partial u^{\alpha}}{\partial u'^{\gamma}} \dfrac{\partial u^{\beta}}{\partial u'^{\delta}}$ .... (2. 11)

$g_{\alpha\beta}$ adalah <u>Komponen - komponen Tensor Matrik Kovarian Pada Suatu Permukaan</u>.

Untuk memperoleh determinan dari $g_{\alpha\beta}$ adalah :

$$ds^2 = g_{11}(du^1)^2 + g_{12}du^1du^2 + g_{21}du^2du^1 + g_{22}(du^2)^2$$

Dalam hal ini kita berbicara tentang Tensor Simetri, maka :

$$g_{12} = g_{21} \quad \text{dan}$$

$$ds^2 = g_{11}(du^1)^2 + 2\, g_{12}\, du^1\, du^2 + g_{22}(du^2)^2$$

Sehingga kita peroleh harga dari :

$$g = |g_{\alpha\beta}| = \begin{vmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{vmatrix}$$

Jika didefinisikan : $g^{\alpha\beta} g_{\beta\gamma} = \delta^{\alpha}_{\gamma}$ .... (2.12)

dimana $\delta^{\alpha}_{\gamma} = 1$, jika $\alpha = \gamma$

$\delta^{\alpha}_{\gamma} = 0$, jika $\alpha \neq \gamma$

Menurut rumus Delta Kronecker yaitu :

$$g^{ik} = \frac{\text{Kofaktor dari } g_{ki} \text{ dalam } g}{g}$$

Maka : $g^{11} = \dfrac{g_{22}}{g}$; $g^{12} = g^{21} = \dfrac{-g_{12}}{g}$; $g^{22} = \dfrac{g_{11}}{g}$

Sesuai dengan persamaan (2.1) kita peroleh :

$g^{\alpha\beta}$ adalah Tensor Matrik Kontravarian Pada Suatu Permukaan. Pandang pula dua fungsi $\lambda^{\alpha}(u^1,u^2)$ dan $\lambda'^{\alpha}(u^1,u^2)$ dalam dua sistem koordinat u^{α} dan u'^{α}, terdapat hubungan :

$$\lambda^{\alpha} = \lambda'^{\beta} \frac{\partial u^{\alpha}}{\partial u'^{\beta}} \quad \text{dan} \quad \lambda'^{\beta} = \lambda^{\beta} \frac{\partial u^{\beta}}{\partial u'^{\alpha}}$$

.... (2.14)

λ^{α} dan λ'^{α} disebut Komponen - komponen Vektor Kontravarian sesuai dengan koordinatnya masing - masing.

Perhatikan pula fungsi - fungsi $\lambda_{\alpha}(u^1,u^2)$ dan $\lambda'_{\alpha}(u^1,u^2)$ dalam dua sistem koordinat λ_{α} dan λ'_{α} yang dinyatakan dengan hubungan :

$$\lambda_{\alpha} = \lambda'_{\beta} \frac{\partial u'^{\beta}}{\partial u^{\alpha}} \quad \text{dan} \quad \lambda'_{\alpha} = \lambda_{\beta} \frac{\partial u^{\beta}}{\partial u'^{\alpha}}$$

...........................92.15)

λ_{α} dan λ'_{α} adalah Komponen - komponen Vektor Kovarian sesuai dengan sistem koordinatnya masing - masing.

Jika λ_{α} adalah kompnen - komponen vektor kovarian , kemudian :

$$\lambda^{\alpha} = g^{\alpha\beta} \lambda_{\beta} \quad \text{.......................... (2.16)}$$

adalah komponen - komponen vektor kontravarian, jika $g^{\alpha\beta}$ merupakan Tensor Kontravarian.

Jika λ^{α} adalah komponen - komponen vektor Kontravarian, kemudian :

$$\lambda_{\alpha} = g_{\alpha\beta} \lambda^{\beta} \quad \text{.......................... (2.17)}$$

adalah komponen - komponen vektor kovarian, jika $g_{\alpha\beta}$ merupakan Tensor Kovarian.

Dua fungsi λ^{α} dan λ_{α} dari u^{α} seperti dalam persamaan (2.16) dan (2.17) adalah komponen - komponen Kontravarian dan komponen - komponen Kovarian dari vektor yang sama.

Didefinisikan besaran :

$$\xi_i = \lambda^\alpha \frac{\partial x^i}{\partial u^\alpha} \quad \text{dan} \quad \xi_i = \lambda'^\beta \frac{\partial x^i}{\partial u'^\beta}$$

.... (2.18)

Vektor λ^α adalah komponen kontravarian, maka setiap besaran skalar $g_{\alpha\beta}\lambda^\alpha\lambda^\beta$ adalah sama dengan kwadrat panjang vektor.

Kita hitung $\sum_i \xi_i \; \xi_i = \sum_i \lambda^\alpha \frac{\partial x^i}{\partial u^\alpha} \; \lambda^\beta \frac{\partial x^i}{\partial u^\beta}$

(dari (2.18))

$$= \lambda^\alpha \lambda^\beta \sum_i \frac{\partial x^i}{\partial u^\alpha} \frac{\partial x^i}{\partial u^\beta}$$

$$= \lambda^\alpha \lambda^\beta g_{\alpha\beta}$$

dimana :

$$g_{\alpha\beta} = \sum_i \frac{\partial x^i}{\partial u^\alpha} \frac{\partial x^i}{\partial u^\beta}$$

Dari (2.16) disubstitusikan ke persamaan :

$$\sum_i \xi_i \; \xi_i = \lambda^\alpha\lambda^\beta \; g_{\alpha\beta} = g^{\alpha\gamma}\lambda_\gamma \, g^{\beta\delta}\lambda_\delta \, g_{\alpha\beta}$$

$$= g^{\alpha\gamma}\lambda_\gamma\lambda_\delta g^{\beta\delta} \; g_{\alpha\beta} = g^{\alpha\gamma}\lambda_\gamma\lambda_\delta \delta^\delta_\alpha$$

dimana $g^{\beta\delta} \; g_{\alpha\beta} = \delta^\delta_\alpha$ (sesuai (2.12))

$$\sum_i \xi_i \; \xi_i = \lambda^\alpha\lambda^\beta g_{\alpha\beta} = g^{\alpha\gamma}\lambda_\gamma \, g^{\beta\delta}\lambda_\delta \, g_{\alpha\beta}$$

$$= g^{\alpha\gamma}\lambda_\gamma\lambda_\delta\delta^\delta_\alpha = g^{\alpha\gamma}\lambda_\gamma\lambda_\delta$$

(sebab $\delta^\delta_\alpha = 1$; (2.12))

Analog di atas, setiap vektor λ_α adalah komponen komponen kovarian, maka setiap besaran skalar $g^{\alpha\beta}\lambda_\alpha\lambda_\beta$ adalah sama dengan kwadrat panjang vektor.

Oleh karena itu syarat perlu dan cukup bahwa λ^α dan λ_α masing - masing adalah komponen kontravarian dan komponen kovarian suatu vektor unit pada suatu permukaan yaitu:

$$g_{\alpha\beta}\lambda^\alpha\lambda^\beta = 1$$

II.2.1. SUDUT YANG DIBENTUK OLEH PERPOTONGAN DUA KURVA.

Didefinisikan τ^i sebagai arah suatu vektor dengan bentuk persamaan : $\tau^i = \dfrac{dx^i}{ds}$

Persamaan (2.3) dan (2.4) disubstitusikan ke dalam persamaan di atas, sehingga :

$$\tau^i = \frac{\dfrac{\partial x^i}{\partial u^\alpha}\, du^\alpha}{\sqrt{g_{\alpha\beta}\; du^\alpha \; du^\beta}} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2.20)$$

Pandang vektor - vektor τ_1^i dan τ_2^j adalah vektor - vektor unit, maka :

$$\begin{vmatrix} \tau_1^i & \tau_2^j \\ \tau_1^i & \tau_2^j \end{vmatrix} = \tau_1^i \;\; \tau_2^j \;\; \text{Cos. } \theta \quad \text{atau} \quad \begin{vmatrix} \tau_1^i & \tau_2^j \\ \tau_1^i & \tau_2^j \end{vmatrix} = \text{Cos. } \theta$$

Maka dapat dihitung :

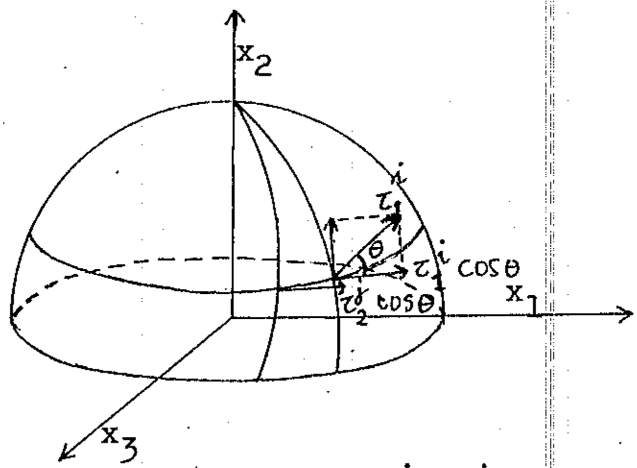
$$\text{Cos. } \theta \times \text{Cos. } \emptyset = \begin{vmatrix} \tau_1^i & \tau_2^j \\ \tau_1^i & \tau_2^j \end{vmatrix} \begin{vmatrix} \tau_1^k & \tau_2^l \\ \tau_1^k & \tau_2^l \end{vmatrix} =$$

$$= \begin{vmatrix} \tau_1^1 & \tau_2^1 \\ \tau_1^1 & \tau_2^1 \end{vmatrix} \begin{vmatrix} \tau_1^1 & \tau_2^1 \\ \tau_1^1 & \tau_2^1 \end{vmatrix} + \begin{vmatrix} \tau_1^1 & \tau_2^1 \\ \tau_1^1 & \tau_2^1 \end{vmatrix} \begin{vmatrix} \tau_1^2 & \tau_2^2 \\ \tau_1^2 & \tau_2^2 \end{vmatrix} +$$

$$+ \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \; + \begin{vmatrix} \tau_1^1 & \tau_2^1 \\ \tau_1^1 & \tau_2^1 \end{vmatrix} \begin{vmatrix} \tau_1^p & \tau_2^q \\ \tau_1^p & \tau_2^q \end{vmatrix} +$$

$$+ \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \; +$$

$$\begin{vmatrix} \tau^r & \tau^s \end{vmatrix} \begin{vmatrix} \tau^p & \tau^q \end{vmatrix}$$



dimana θ adalah sudut yang dibentuk oleh dua buah vektor unit yang terdapat pada suatu permukaan. Untuk θ sama dengan $\emptyset$, maka :

$$\operatorname{Cos}^2 \theta = \sum_i \tau_1^i \, \tau_2^i = \sum_i \frac{\frac{\partial x^i}{\partial u^\alpha} \, d_1 u^\alpha}{\sqrt{g_{\alpha\beta} \, d_1 u^\alpha \, d_1 u^\beta}} \, \frac{\frac{\partial x^i}{\partial u^\beta} \, d_2 u^\beta}{\sqrt{g_{\gamma\delta} \, d_2 u^\gamma \, d_2 u^\delta}}$$

$$= \frac{d_1 u^\alpha \quad d_2 u^\beta}{\sqrt{g_{\alpha\beta} \, d_1 u^\alpha \, d_1 u^\beta} \, \sqrt{g_{\gamma\delta} \, d_2 u^\gamma \, d_2 u^\delta}} \sum_i \frac{\partial x^i}{\partial u^\alpha} \, \frac{\partial x^i}{\partial u^\beta}$$

$$\operatorname{Cos}^2 \theta = \frac{g_{\alpha\beta} \, d_1 u^\alpha \, d_2 u^\beta}{\sqrt{g_{\alpha\beta} \, d_1 u^\alpha \, d_1 u^\beta} \, \sqrt{g_{\gamma\delta} \, d_2 u^\gamma \, d_2 u^\delta}} \quad \ldots\ldots\ldots(2.21)$$

dan dengan demikian, maka :

$$\operatorname{Sin}^2 \theta = 1 - \operatorname{Cos}^2 \theta$$

$$= \begin{vmatrix} \sum_i \tau_1^i \, \tau_1^i & \sum_i \tau_1^i \, \tau_2^i \\ \sum_i \tau_1^i \, \tau_2^i & \sum_i \tau_2^i \, \tau_2^i \end{vmatrix}^2$$

$$= \begin{vmatrix} \tau_1^1 & \tau_1^2 \\ \tau_2^1 & \tau_2^2 \end{vmatrix}^2 + \begin{vmatrix} \tau_1^2 & \tau_1^3 \\ \tau_2^2 & \tau_2^3 \end{vmatrix}^2 + \begin{vmatrix} \tau_1^3 & \tau_1^1 \\ \tau_2^3 & \tau_2^1 \end{vmatrix}^2$$

(menurut Rumus Binormal)

Dengan mengingat (2.20) terlebih dahulu kita mencari harga :

$$\begin{vmatrix} \tau_1^i & \tau_1^j \\ & \end{vmatrix} = \begin{vmatrix} \frac{\partial x^i}{\partial u^\alpha} \frac{d_1 u^\alpha}{d_1 s} & \frac{\partial x^j}{\partial u^\alpha} \frac{d_1 u^\alpha}{d_1 s} \\ & \end{vmatrix}$$

$$\begin{vmatrix} \tau_1^i & \tau_1^j \\ \tau_2^i & \tau_2^j \end{vmatrix} = \begin{vmatrix} \frac{\partial x^i}{\partial u^1} & \frac{\partial x^j}{\partial u^1} \\ \frac{\partial x^i}{\partial u^2} & \frac{\partial x^j}{\partial u^2} \end{vmatrix} \begin{vmatrix} \frac{d_1 u^1}{d_1 s} & \frac{d_1 u^1}{d_1 s} \\ \frac{d_2 u^1}{d_2 s} & \frac{d_2 u^2}{d_2 s} \end{vmatrix}$$

Maka kita peroleh :

$$\sin^2 \theta = \begin{vmatrix} \frac{\partial x^1}{\partial u^1} & \frac{\partial x^2}{\partial u^1} \\ \frac{\partial x^1}{\partial u^2} & \frac{\partial x^2}{\partial u^2} \end{vmatrix}^2 \begin{vmatrix} \frac{d_1 u^1}{d_1 s} & \frac{d_1 u^2}{d_1 s} \\ \frac{d_2 u^1}{d_2 s} & \frac{d_2 u^2}{d_2 s} \end{vmatrix}^2 + \begin{vmatrix} \frac{\partial x^2}{\partial u^1} & \frac{\partial x^3}{\partial u^1} \\ \frac{\partial x^2}{\partial u^2} & \frac{\partial x^3}{\partial u^2} \end{vmatrix}^2 \begin{vmatrix} \frac{d_1 u^1}{d_1 s} & \frac{d_1 u^2}{d_1 s} \\ \frac{d_2 u^1}{d_2 s} & \frac{d_2 u^2}{d_2 s} \end{vmatrix}^2 + \begin{vmatrix} \frac{\partial x^3}{\partial u^1} & \frac{\partial x^1}{\partial u^1} \\ \frac{\partial x^3}{\partial u^2} & \frac{\partial x^1}{\partial u^2} \end{vmatrix}^2 \begin{vmatrix} \frac{d_1 u^1}{d_1 s} & \frac{d_1 u^2}{d_1 s} \\ \frac{d_2 u^1}{d_2 s} & \frac{d_2 u^2}{d_2 s} \end{vmatrix}^2$$

.... (2.22)

Mengingat definisi :

$$A^{ij} = \frac{\partial(f^i, f^j)}{\partial(u^1, u^2)} = \begin{vmatrix} \frac{\partial f^i}{\partial u^1} & \frac{\partial f^j}{\partial u^1} \\ \frac{\partial f^i}{\partial u^2} & \frac{\partial f^j}{\partial u^2} \end{vmatrix}$$ dengan;

$i, j = 1, 2, 3;\ i \neq j$ .... (2.23)

Maka persamaan (2.21) menjadi :

$$\sin^2 \theta = (A^{12})^2 + (A^{23})^2 + (A^{31})^2 \begin{vmatrix} \frac{d_1 u^1}{d_1 s} & \frac{d_1 u^2}{d_1 s} \\ \frac{d_2 u^1}{d_2 s} & \frac{d_2 u^2}{d_2 s} \end{vmatrix}^2$$

Telah didefinisikan bahwa :

$$g = \begin{vmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{vmatrix} = |g_{\alpha\beta}| = (A^{12})^2 + (A^{23})^2 + (A^{31})^2$$

Maka persamaan (2.23) menjadi :

$$\sin^2\theta = g \begin{vmatrix} \frac{d_1u^1}{d_1s} & \frac{d_2u^2}{d_1s} \\ \frac{d_2u^1}{d_2s} & \frac{d_2u^2}{d_2s} \end{vmatrix}^2$$

$$= g \left(\frac{d_1u^1}{d_1s}\,\frac{d_2u^2}{d_2s} - \frac{d_2u^1}{d_1s}\,\frac{d_1u^2}{d_2s} \right)^2$$

$$= g \left(\frac{d_1u^1\ d_2u^2 - d_1u^2\ d_2u^1}{\sqrt{(g_{\alpha\beta}\ d_1u^\alpha\ d_1u^\beta)(g_{\gamma\delta}\ d_2u^\gamma\ d_2u^\delta)}} \right)^2$$

$$\sin\theta = \sqrt{g} \left(\frac{d_1u^1\ d_2u^2 \qquad d_2u^1\ d_1u^2}{\sqrt{(g_{\alpha\beta}\ d_1u^\alpha\ d_1u^\beta)(g_{\gamma\delta}\ d_2u^\gamma\ d_2u^\delta)}} \right)$$

..................................(2.25)

TEOREMA 2.1 :

Syarat perlu dan cukup bahwa pada suatu titik, vektor - vektor $\lambda_{1|}^{\alpha}$ dan $\lambda_{2|}^{\beta}$ saling tegak lurus, adalah : $g^{\alpha\beta}\ \lambda_{1|\alpha}\ \lambda_{2|\beta} = 0$ dimana $\lambda_{1|\alpha}$ dan $\lambda_{2|\beta}$ adalah komponen - komponen kovarian dari masing - masing vektor.

Bukti :

Kita pandang dua komponen vektor kontravarian $\lambda_{1|}^{\alpha}$ dan $\lambda_{2|}^{\beta}$

Jika pada suatu titik dari suatu permukaan diperoleh diferensial d_1u^α dan d_2u^β yang disajikan dengan persamaan:

$$d_1 u^{\alpha} = \lambda^{\alpha}_{1|} \quad \text{dan} \quad d_2 u^{\alpha} = \lambda^{\beta}_{2|}$$

Maka persamaan (2.21) dan (2.25) menjadi :

$$\text{Cos. } \Theta = \frac{g_{\alpha\beta}\, \lambda^{\alpha}_{1|}\ \lambda^{\beta}_{2|}}{(g_{\alpha\beta}\, \lambda^{\alpha}_{1|}\ \lambda^{\beta}_{1|})(g_{\gamma\delta}\, \lambda^{\gamma}_{2|}\ \lambda^{\delta}_{2|})}$$

$$\text{Sin. } \Theta = \frac{\lambda^{1}_{1|}\ \lambda^{2}_{2|} - \lambda^{2}_{1|}\ \lambda^{1}_{2|}}{(g_{\alpha\beta}\, \lambda^{\alpha}_{1|}\ \lambda^{\beta}_{1|})(g_{\gamma\delta}\, \lambda^{\gamma}_{2|}\ \lambda^{\delta}_{2|})}$$

....................................(2.26)

diama Θ adalah sudut yang dibentuk oleh $\lambda^{\alpha}_{1|}$ dan $\lambda^{\beta}_{2|}$

$$g_{\alpha\beta}\, \lambda^{\alpha}_{1|}\ \lambda^{\beta}_{2|} = g_{\alpha\beta}\ g^{\alpha\gamma} \lambda_{1|\gamma}\ g^{\beta\delta} \lambda_{2|\delta}$$

(sesuai (2.16))

$$= g^{\alpha\gamma}\, g_{\alpha\beta}\, g^{\beta\delta}\, \lambda_{1|\gamma}\, \lambda_{2|\delta}$$

$$= g^{\alpha\gamma}\, \delta^{\delta}_{\alpha}\, \lambda_{1|\gamma}\, \lambda_{2|\delta}$$

(sesuai (2.12))

$$= g^{\alpha\gamma}\, \lambda_{1|\gamma}\ \lambda_{2|\delta} \qquad (\text{ sebab } \delta^{\delta}_{\alpha} = 1)$$

$$\lambda^{1}_{1|}\ \lambda^{2}_{2|} - \lambda^{2}_{1|}\ \lambda^{1}_{2|} = g^{1\alpha}\ \lambda_{1|\alpha}\, g^{2\beta} \lambda_{2|\beta} - g^{2\alpha} \lambda_{1|\alpha}\, g^{1\beta} \lambda_{2|\beta}$$

(sesuai (2.16))

$$= g^{1\alpha}\, g^{2\beta} \lambda_{1|\alpha}\, \lambda_{2|\beta} - g^{2\alpha}\, g^{1\beta} \lambda_{1|\alpha}\, \lambda_{2|\beta}$$

$$= g^{11}\, g^{22} \lambda_{1|1} \lambda_{2|2} - g^{12}\, g^{21} \lambda_{1|2} \lambda_{2|1} +$$

$$- g^{21}\, g^{12} \lambda_{1|1} \lambda_{2|2} - g^{22}\, g^{11} \lambda_{1|2} \lambda_{2|1}$$

$$\lambda^{1}_{1|}\ \lambda^{2}_{2|} - \lambda^{2}_{1}\ \lambda^{1}_{2} = g^{11}\, g^{22} (\lambda_{1|1} \lambda_{2|2} - \lambda_{1|2} \lambda_{2|1}) +$$

$$- g^{12}\, g^{12} (\lambda_{1|1} \lambda_{2|2} - \lambda_{1|2} \lambda_{2|1})$$

$$= (g^{11}\, g^{22} - (g^{12})^2)(\lambda_{1|1} \lambda_{2|2} - \lambda_{1|2} \lambda_{2|1})$$

$$= g\ (\lambda_{1|1} \lambda_{2|2} - \lambda_{1|2} \lambda_{2|1})$$

Maka persamaan (2,26) menjadi :

$$\text{Cos. } \theta = \frac{g^{\alpha\beta} \lambda_{1|\alpha} \lambda_{2|\beta}}{\sqrt{(g^{\alpha\beta} \lambda_{1|\alpha} \lambda_{1|\beta})(g^{\gamma\delta} \lambda_{2|\gamma} \lambda_{2|\delta})}}$$

$$\text{Sin. } \theta = \frac{1}{\sqrt{g}} \frac{\lambda_{1|1} \lambda_{2|2} - \lambda_{1|2} \lambda_{2|1}}{\sqrt{(g^{\alpha\beta}\lambda_{1|\alpha} \lambda_{1|\beta})(g^{\gamma\delta}\lambda_{2|\alpha} \lambda_{2|\beta})}}$$

Telah ditentukan bahwa $\lambda^{\alpha}_{1|}$ dan $\lambda^{\beta}_{2|}$ saling tegak lurus

maka, Cos. $\theta = 0$; artinya $\theta = 90^{o}$ atau :

$$\frac{g^{\alpha\beta} \lambda_{1|\alpha} \lambda_{2|\beta}}{(g^{\alpha\beta}\lambda_{1|\alpha} \lambda_{1|\beta})(g^{\gamma\delta} \lambda_{2|\gamma} \lambda_{2|\delta})} = 0$$

$$g^{\alpha\beta} \lambda_{1|\alpha} \lambda_{2|\beta} = 0 \qquad (\text{ sesuai } (2.17) \text{ }) \text{ },$$

dan mengingat :

$$g_{\alpha\beta} \lambda^{\alpha}_{1|} \lambda^{\beta}_{2|} = \lambda_{1|\beta} \lambda^{\beta}_{2|} = \lambda^{\alpha}_{1|} \lambda^{\beta}_{2|} g^{\alpha\beta}$$

$$= \lambda^{\alpha}_{1|} \lambda_{2|\alpha} = g^{\alpha\beta} \lambda_{1|\alpha} \lambda_{2|\beta}$$

Sehingga terbukti bahwa :

$$g_{\alpha\beta} \lambda^{\alpha}_{1|} \lambda^{\beta}_{2|} = g^{\alpha\beta} \lambda_{1|\alpha} \lambda_{2|\beta} = 0$$

(maka Teorema di atas telah terbukti)

II.2.2. KOMPONEN - KOMPONEN KONTRAVARIAN DAN KOVARIAN.

Didefinisikan :

Dua himpunan bilangan $e_{\alpha\beta}$ dan $e^{\alpha\beta}$ seperti :

$$e_{11} = e_{22} = e^{11} = e^{22} = 0$$

$$e_{12} = e^{12} = 1 \qquad \ldots\ldots\ldots\ldots\ldots \quad (2.27)$$

$$e_{21} = e^{21} = -1$$

Untuk transformasi koordinat pada suatu permukaan terdapat-

persamaan :

$$e_{\alpha\beta} \frac{\partial u^\alpha}{\partial u'^1} \frac{\partial u^\beta}{\partial u'^1} = e_{\alpha\beta} \frac{\partial u^\alpha}{\partial u'^2} \frac{\partial u^\beta}{\partial u'^2} = 0$$

$$e_{\alpha\beta} \frac{\partial u^\alpha}{\partial u'^1} \frac{\partial u^\beta}{\partial u'^2} = \frac{\partial (u^1, u^2)}{\partial (u'^1, u'^2)} = -e_{\alpha\beta} \frac{\partial u^\alpha}{\partial u'^2} \frac{\partial u^\beta}{\partial u'^1}$$

Jika transformasi positip, didefinisikan :

$$\sqrt{g'} = \sqrt{g} \; \frac{\partial (u^1, u^2)}{\partial (u'^1, u'^2)} \quad \ldots\ldots\ldots\ldots\ldots\ldots \; (2.28)$$

Maka :

$$e_{\gamma\delta} \sqrt{g'} = e_{\alpha\beta} \sqrt{g'} \qquad (\text{ menurut } (2.27) \text{ })$$

$$= e_{\alpha\beta} \sqrt{g} \; \frac{\partial u^\alpha}{\partial u'^\gamma} \frac{\partial u^\beta}{\partial u'^\delta}$$

$$\frac{e^{\gamma\delta}}{\sqrt{g'}} = \frac{e^{\alpha\beta}}{\sqrt{g'}} = \frac{e^{\alpha\beta}}{\sqrt{g}} \; \frac{u^\alpha}{u'^\gamma} \; \frac{u^\beta}{u'^\delta}$$

Transformasi positip dari Skew Simetri, besarnya - komponen - komponen kovarian dan komponen - komponen kontra varian orde dua berturut - turut disajikan seperti berikut:

$$\varepsilon_{\alpha\beta} = e_{\alpha\beta} \sqrt{g} \quad \text{dan} \quad \varepsilon^{\alpha\beta} = \frac{e^{\alpha\beta}}{\sqrt{g}}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \; (2.29)$$

TEOREMA 2.2 :

Syarat perlu dan cukup bahwa suatu vektor unit $\lambda^\beta_{2|}$ membentuk suatu sudut siku - siku dengan vektor - unit $\lambda^\alpha_{1|}$ adalah :

$$\varepsilon_{\alpha\beta} \, \lambda^\alpha_{1|} \, \lambda^\beta_{2|} = 1$$

Bukti :

Diambil bentuk persamaan $\varepsilon^{\alpha\gamma}\varepsilon_{\beta\gamma} = \varepsilon^{\gamma\alpha}\varepsilon_{\gamma\beta}$

Dari bentuk persamaan (2.29), maka :

$$\varepsilon^{\alpha\gamma}\varepsilon_{\beta\gamma} = \frac{e^{\alpha\gamma}}{g} e_{\beta\gamma}\, g = e^{\alpha\gamma} e_{\beta\gamma} = \delta^{\alpha}_{\beta}$$

(sesuai (2.12))

$$= \varepsilon^{\gamma\alpha}\varepsilon_{\gamma\beta} = \delta^{\alpha}_{\beta}$$

Kembali kita perhatikan bentuk :

$$\text{Sin. } \Theta = \frac{\frac{1}{\sqrt{g}}\left(\lambda^{1}_{1|}\lambda^{2}_{2|} - \lambda^{1}_{2|}\lambda^{2}_{1|}\right)}{\sqrt{(g_{\alpha\beta}\lambda^{\alpha}_{1|}\lambda^{\beta}_{1|})(g_{\gamma\delta}\lambda^{\gamma}_{2|}\lambda^{\delta}_{2|})}}$$

Disajikan dalam bentuk :

$$\text{Sin. } \Theta = \frac{\varepsilon_{\alpha\beta}\lambda^{\alpha}_{1|}\lambda^{\beta}_{2|}}{\sqrt{(g_{\alpha\beta}\lambda^{\alpha}_{1|}\lambda^{\beta}_{1|})(g_{\gamma\delta}\lambda^{\gamma}_{2|}\lambda^{\delta}_{2|})}}$$

Telah disyaratkan bahwa $\lambda^{\alpha}_{1|}$ dan $\lambda^{\beta}_{2|}$ membentuk sudut siku - siku, maka :

Sin. Θ = 1 atau;

$$\frac{\varepsilon_{\alpha\beta}\lambda^{\alpha}_{1|}\lambda^{\beta}_{2|}}{\sqrt{(g_{\alpha\beta}\lambda^{\alpha}_{1|}\lambda^{\beta}_{1|})(g_{\gamma\delta}\lambda^{\gamma}_{2|}\lambda^{\delta}_{2|})}} = 1$$

Jadi : $\varepsilon_{\alpha\beta}\lambda^{\alpha}_{1|}\lambda^{\beta}_{2|} = 1$, sebab $\lambda_{1|}, \lambda_{2|}$ vektor - unit.

Maka Teorema di atas telah terbukti.

TEOREMA 2.3 :

Didefinisikan $\mu_{\alpha} = \varepsilon_{\alpha\beta}\lambda^{\beta}$ adalah komponen kovarian dan μ_{α}, membentuk sudut siku - siku dengan vektor

<u>Bukti</u> :

Menurut Teorema 2.1, jika $\mu_\alpha \lambda^\alpha = 0$, maka μ_α tegak lurus pada vektor λ^α.

Komponen - komponen kontravarian diberikan oleh :

$$\mu^\alpha = g^{\alpha\beta}\mu_\beta = g^{\alpha\beta}\varepsilon_{\alpha\beta}\lambda^\gamma \quad (\text{ menurut } (2.16)\ )$$

$$= g^{\alpha 2}\lambda^1 \varepsilon_{12} + g^{\alpha 1}\lambda^2 \varepsilon_{21} \quad (\text{ sesuai } (2.30)\ )$$

$$= g^{\alpha 2}\lambda^1 e_{12}\sqrt{g} + g^{\alpha 1}\lambda^2 e_{21}\sqrt{g}$$

(menurut (2.29))

$$= g^{\alpha 2}\lambda^1\, 1\sqrt{g} + g^{\alpha 1}\lambda^2 (-1)\sqrt{g}$$

(menurut (2.27))

$$= (g^{\alpha 2}\lambda^1 - g^{\alpha 1}\lambda^2)\sqrt{g}$$

$$= (g^{\alpha 2} g^{\beta 1}\lambda_\beta - g^{\alpha 1} g^{\beta 2}\lambda_\beta)\sqrt{g}$$

$$\mu^\alpha = \varepsilon^{\beta\alpha}\lambda_\beta \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.31)$$

Maka : $\mu^\alpha \mu_\alpha = \varepsilon^{\beta\alpha}\lambda_\beta \cdot \varepsilon_{\beta\alpha}\lambda^\beta$

$$= \varepsilon^{\beta\alpha}\varepsilon_{\beta\alpha}\lambda_\beta\lambda^\beta = \lambda_\beta\lambda^\beta$$

Menurut Teorema 2.2, maka terlihat bahwa :

$$\mu^\alpha \mu_\alpha = \lambda_\beta \lambda^\beta = 1$$

Mengingat pula bahwa :

$$\varepsilon^{\alpha\gamma}\varepsilon_{\beta\gamma} = \varepsilon^{\gamma\alpha}\varepsilon_{\gamma\beta} = \delta^\alpha_\beta \quad , \text{ maka } :$$

$$\varepsilon_{\alpha\beta}\lambda^\alpha\mu^\beta = \varepsilon_{\alpha\beta}\lambda^\alpha\varepsilon^{\alpha\beta}\lambda_\alpha = \varepsilon_{\alpha\beta}\varepsilon^{\alpha\beta}\lambda^\alpha\lambda_\alpha$$

$$= 1$$

Maka terbukti komponen - komponen kovarian μ_α membentuk sudut siku - siku dengan vektor unit λ^α.

II.3. FAMILY KURVA PADA SUATU PERMUKAAN.

Di atas suatu permukaan didefinisikan persamaan suatu kurva :

$$f\,(u^1,u^2) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.32)$$

dimana u^α adalah batas - batas koordinat, $\alpha = 1,2$.
Jika kurva tersebut didefinisikan pula sebagai fungsi dari parameter t, yaitu :

$$u^\alpha = Q^\alpha(t) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.33)$$

Sehingga dari persamaan (2.32) dan (2.33) jika diturunkan, diperoleh :

$$\frac{\partial f}{\partial u^\alpha}\,\frac{du^\alpha}{dt} = 0 \quad \text{atau} \quad \frac{\partial f}{\partial u^\alpha}\,\frac{dQ}{dt} = 0$$

Pandang persamaan : $f\,(u^1,u^2) = c$ .... (2.34)
dimana c = konstan.
Persamaan ini sesuai dengan persamaan (2.1), dimana jika diambil f bernilai tunggal, maka menjadi persamaan (2.6).
Persamaan (2.6) ini menjadi TITIK TEMBUS SUATU KURVA, jika c diberikan harga yang khusus.

Didefinisikan persamaan $\dfrac{\partial f}{\partial u^\alpha}\, du^\alpha = 0$ ....(2.35)

Diambil $f_1(u^1,u^2) = F(\,f(u^1,u^2)\,)$. Maka tempat kedudukan titik - titik untuk f adalah konstanta, merupakan pula suatu tempat kedudukan titik - titik untuk f_1 adalah konstan.

Sebaliknya jika family kurva dari $f_1(u^1,u^2) = c_1$ adalah family kurva dari $f\,(u^1,u^2) = c$, maka f_1 adalah fungsi dari f. du^α menentukan arah kurva pada setiap family kurva tersebut, sehingga :

$$\frac{\partial f}{\partial u^\alpha}\, du^\alpha = 0 \quad ; \quad \frac{\partial f_1}{\partial u^\alpha}\, du^\alpha = 0$$

Diambil persamaan :

$$M_\alpha \; du^\alpha = 0 \; ; \; \alpha = 1,2 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2.36)$$

dimana M_1 dan M_2 adalah fungsi dari u .

Didefinisikan persamaan faktor integral :

$$\frac{\partial f}{\partial u^\alpha} = t \, M_\alpha \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2.37)$$

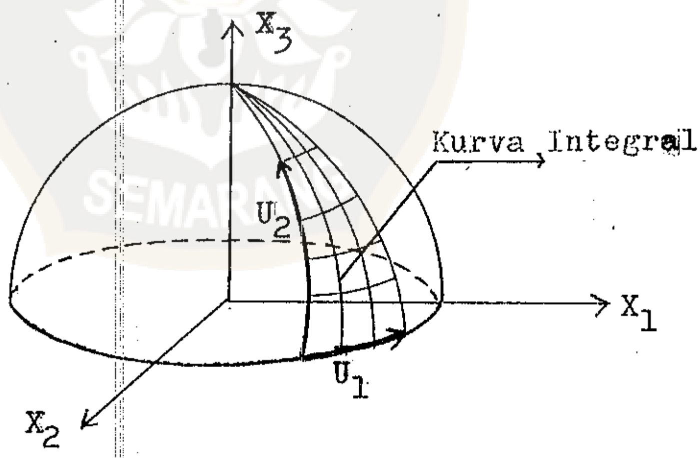
atau :

$$t \; M_\alpha \; du = df$$

<u>Definisi</u> :

Jika terdapat satu family kurva pada suatu permukaan, maka family kurva itu disebut suatu KURVA INTEGRAL.

Disajikan dalam bentuk gambar :



<u>TEOREMA 2.4</u> :

Trayektori Ortogonal dari suatu kurva integral dengan persamaan $M_\alpha \; du^\alpha = 0$ adalah integral dari persamaan :

$$(g_{11} \, M_2 - g_{12} \, M_1) \, du^1 + (g_{12} \, M_2 - g_{22} \, M_1) \, du^2 = 0$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2.38)$$

<u>Bukti</u> :

Diambil $du^1 = p \, M_2$ dan $du^2 = -p \, M_1$

persamaan - persamaan ini dipenuhi untuk harga p sebarang.

Jika d_2u^1 dan d_2u^2 kita substitusikan ke dalam persamaan tersebut, maka :

$$d_2u^1 = p\,M_2 \quad \text{dan} \quad d_2u^2 = -p\,M_1$$

Persamaan - persamaan ini disubstitusikan ke $g_{\alpha\beta}\, d_2u^\alpha\, d_1u^\beta = 0$ (mengingat vektor d_2u^α dan d_1u^β saling tegak lurus) , maka :

$$g_{\alpha\beta}\, d_2u^\alpha\, d_1u^\beta = 0$$

$$g_{1\beta}\, d_2u^1\, d_1u^\beta + g_{2\beta}\, d_2u^2\, d_1u^\beta = 0$$

$$g_{1\beta}\,(\, p\,M_2\,)\, d_1u^\beta + g_{2\beta}\,(\, -p\,M_1\,)\, d_1u^\beta = 0$$

$$(\, g_{1\beta}\,M_2 - g_{2\beta}\,M_1\,)\, d_1u^\beta = 0$$

Maka kita peroleh persamaan :

$$(\, g_{11}\,M_2 - g_{21}\,M_1\,)\, d_1u^1 + (\, g_{12}\,M_2 - g_{22}\,M_1\,)\, d_1u^2 = 0$$

$$(\, g_{11}\,M_2 - g_{12}\,M_1\,)\, du^1 + (\, g_{12}\,M_2 - g_{22}\,M_1\,)\, du^2 = 0$$

Sehingga terbuktilah teorema di atas.

AKIBAT TEOREMA :

*). Jika $M_2 = 0$, trayektori ortogonal dari kurva koordinat u^1 = konstan adalah kurva integral - dari persamaan :

$$g_{12}\, du^1 + g_{22}\, du^2 = 0$$

**). Jika $M_1 = 0$, trayektori ortogonal dari kurva koordinat u^2 = konstan adalah kurva integral- dari persamaan :

$$g_{11}\, du^1 + g_{12}\, du^2 = 0$$

Definisi :

Definisi :

Dua family kurva integral dari persamaan :

$a_{\alpha\beta}\ du^{\alpha}\ du^{\beta} = 0$ membentuk ortogonal net bila dan hanya bila :

$$g_{11}\, a_{22} - 2\, g_{12}\, a_{12} + g_{22}\, a_{11} = 0$$

Akan kita buktikan bahwa dua vektor pada suatu titik saling tegak lurus, demikian :

Pandang persamaan :

$$(a_{\alpha\beta} - r\, g_{\alpha\beta})\, \lambda^{\beta} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (2.39)$$

Jika harga $a_{\alpha\beta}$ tidak sama dengan perbanyakan dari $g_{\alpha\beta}$ dengan suatu konstanta tertentu, maka tensor $a_{\alpha\beta}$ tidak sebanding dengan tensor $g_{\alpha\beta}$. Persamaan (2.39) mempunyai penyelesaian $\lambda^{\beta} = 0$, jika tidak demikian r adalah akar persamaan (2.39).

Ambil : $\left| a_{\alpha\beta} - r\, g_{\alpha\beta} \right| = 0$

Misalkan r_1 dan r_2 adalah akar dari persamaan tersebut $\lambda^{\beta}_{1|}$ dan $\lambda^{\beta}_{2|}$ adalah vektor - vektor yang bersesuaian, berturut - turut terhadap r_1 dan r_2.

Maka kita mempunyai bentuk persamaan sebagai berikut :

$$(a_{\alpha\beta} - r_1\, g_{\alpha\beta})\, \lambda^{\beta}_{1|} = 0$$

$$(a_{\alpha\beta} - r_2\, g_{\alpha\beta})\, \lambda^{\beta}_{2|} = 0$$

kemudian masing - masing digandakan dengan $\lambda^{\alpha}_{2|}$ dan $\lambda^{\alpha}_{1|}$, sehingga :

$$a_{\alpha\beta}\, \lambda^{\alpha}_{2|}\, \lambda^{\beta}_{1|} - r_1\, g_{\alpha\beta}\, \lambda^{\alpha}_{2|}\, \lambda^{\beta}_{1|} = 0$$

$$a_{\alpha\beta}\, \lambda^{\alpha}_{1|}\, \lambda^{\beta}_{2|} - r_2\, g_{\alpha\beta}\, \lambda^{\alpha}_{1|}\, \lambda^{\beta}_{2|} = 0$$

$$
\begin{array}{l}
a_{\alpha\beta}\,\lambda_{1|}^{\alpha}\,\lambda_{2|}^{\beta} - r_1\, g_{\alpha\beta}\,\lambda_{1|}^{\alpha}\,\lambda_{2|}^{\beta} = 0 \\
a_{\alpha\beta}\,\lambda_{1|}^{\alpha}\,\lambda_{2|}^{\beta} - r_2\, g_{\alpha\beta}\,\lambda_{1|}^{\alpha}\,\lambda_{2|}^{\beta} = 0 \quad - \\
\hline
(r_2 - r_1)\, g_{\alpha\beta}\,\lambda_{1|}^{\alpha}\,\lambda_{2|}^{\beta} = 0
\end{array}
$$

Jika $r_1 \neq r_2$, kita peroleh :

$$g_{\alpha\beta}\,\lambda_{1|}^{\alpha}\,\lambda_{2|}^{\beta} = 0$$

maka : $g_{\alpha\beta}\,\lambda_{1|}^{\alpha}\,\lambda_{2|}^{\beta} = 0$ .... (2.40)

(memenuhi Teorema 2.1)

Jadi telah terbukti bahwa dua vektor pada suatu titik saling tegak lurus.

II.4. SIMBOL - SIMBOL CHRISTOFFEL UNTUK SUATU PERMUKAAN.

Christoffel mendefinisikan simbol - simbolnya dengan notasi :

$$[ij,k] = \tfrac{1}{2}\left(\frac{\partial g_{ik}}{\partial u^j} + \frac{\partial g_{jk}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^k} \right)$$

$$\left\{ \begin{matrix} h \\ i\;j \end{matrix} \right\} = g^{hk}\,[ij,k] \qquad \text{.... (2.41)}$$

Kita menurunkan simbol - simbol Christoffel tersebut,dengan cara sebagai berikut :

$$[\alpha\alpha\,,\alpha] = \tfrac{1}{2}\left(\frac{\partial g_{\alpha\alpha}}{\partial u^{\alpha}} + \frac{\partial g_{\alpha\alpha}}{\partial u^{\alpha}} - \frac{\partial g_{\alpha\alpha}}{\partial u^{\alpha}} \right) = \tfrac{1}{2}\,\frac{\partial g_{\alpha\alpha}}{\partial u^{\alpha}}$$

$$[\alpha\alpha\,,\beta] = \tfrac{1}{2}\left(\frac{\partial g_{\alpha\beta}}{\partial u^{\alpha}} + \frac{\partial g_{\alpha\beta}}{\partial u^{\alpha}} - \frac{\partial g_{\alpha\alpha}}{\partial u^{\beta}} \right)$$

$$= \tfrac{1}{2}\left(2\,\frac{\partial g_{\alpha\beta}}{\partial u^{\alpha}} - \frac{\partial g_{\alpha\alpha}}{\partial u^{\beta}} \right)$$

$$[\alpha\beta\,,\alpha] = \tfrac{1}{2}\left(\frac{\partial g_{\alpha\beta}}{\partial u^{\alpha}} + \frac{\partial g_{\alpha\alpha}}{\partial u^{\beta}} - \frac{\partial g_{\alpha\beta}}{\partial u^{\alpha}} \right) = \tfrac{1}{2}\,\frac{\partial g_{\alpha\alpha}}{\partial u^{\beta}}$$

$$[\alpha\beta\,,\beta] = \tfrac{1}{2}\left(\frac{\partial g_{\beta\beta}}{\partial u^{\alpha}} + \frac{\partial g_{\alpha\beta}}{\partial u^{\beta}} - \frac{\partial g_{\alpha\beta}}{\partial u^{\beta}}\right) = \tfrac{1}{2}\,\frac{\partial g_{\beta\beta}}{\partial u^{\alpha}}$$

Mengingat definisi :

$$g^{11} = \frac{g_{22}}{g} \quad ; \quad g^{12} = g^{21} = -\frac{g_{12}}{g} \quad ; \quad g^{22} = \frac{g_{11}}{g}$$

Maka persamaan - persamaan di bawah ini :

$$\left\{ {\alpha \atop \alpha\ \alpha} \right\} = g^{\alpha\alpha}[\alpha\alpha,\alpha] + g^{\alpha\beta}[\alpha\alpha,\beta]$$

$$= g^{\alpha\alpha}\left(\tfrac{1}{2}\frac{\partial g_{\alpha\alpha}}{\partial u^{\alpha}}\right) + g^{\alpha\beta}\left(\tfrac{1}{2}\left(2\frac{g_{\alpha\beta}}{\partial u^{\alpha}} - \frac{\partial g_{\alpha\alpha}}{\partial u^{\beta}}\right)\right)$$

$$= \tfrac{1}{2}\left(g^{\alpha\alpha}\frac{\partial g_{\alpha\alpha}}{\partial u^{\alpha}} + \frac{\partial g_{\alpha\beta}}{\partial u^{\alpha}}\,2\,g^{\alpha\beta} - g^{\alpha\beta}\frac{\partial g_{\alpha\alpha}}{\partial u^{\beta}}\right)$$

$$\left\{ {\alpha \atop \alpha\ \alpha} \right\} = \tfrac{1}{2}\left(g^{\alpha\alpha}\frac{\partial g_{\alpha\alpha}}{\partial u^{\alpha}} + 2\,g^{\alpha\beta}\frac{\partial g_{\alpha\beta}}{\partial u^{\alpha}} - g^{\alpha\beta}\frac{\partial g_{\alpha\alpha}}{\partial u^{\beta}}\right)$$

$$\left\{ {\beta \atop \alpha\ \alpha} \right\} = g^{\beta\alpha}[\alpha\alpha,\alpha] + g^{\beta\beta}[\alpha\alpha,\beta]$$

$$= g^{\alpha\beta}\left(\tfrac{1}{2}\frac{\partial g_{\alpha\alpha}}{\partial u^{\alpha}}\right) + g^{\beta\beta}\left(\tfrac{1}{2}\left(2\frac{\partial g_{\alpha\beta}}{\partial u^{\alpha}} - \frac{\partial g_{\alpha\alpha}}{\partial u^{\beta}}\right)\right)$$

$$= \tfrac{1}{2}\left(g^{\alpha\beta}\frac{\partial g_{\alpha\alpha}}{\partial u^{\alpha}} + 2\,g^{\beta\beta}\frac{\partial g_{\alpha\beta}}{\partial u^{\alpha}} - g^{\beta\beta}\frac{\partial g_{\alpha\alpha}}{\partial u^{\beta}}\right)$$

$$\left\{ {\alpha \atop \alpha\ \beta} \right\} = g^{\alpha\alpha}[\alpha\beta,\alpha] + g^{\alpha\beta}[\alpha\beta,\beta]$$

$$= g^{\alpha\alpha}\left(\tfrac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial u^{\beta}}\right) + g^{\alpha\beta}\left(\tfrac{1}{2}\frac{\partial g_{\beta\beta}}{\partial u^{\alpha}}\right)$$

$$= \tfrac{1}{2}\left(g^{\alpha\alpha}\frac{\partial g_{\alpha\beta}}{\partial u^{\beta}} - g^{\alpha\beta}\frac{\partial g_{\beta\beta}}{\partial u^{\alpha}}\right)$$

menjadi :

$$\left\{ {\alpha \atop \alpha\ \alpha} \right\} = \frac{1}{2g}\left(g_{\beta\beta}\frac{\partial g_{\alpha\alpha}}{\partial u^{\alpha}} - 2\,g_{\alpha\beta}\frac{\partial g_{\alpha\beta}}{\partial u^{\beta}} + g_{\alpha\beta}\frac{\partial g_{\alpha\alpha}}{\partial u^{\beta}}\right)$$

.... (2.42)

$$\left\{ \begin{matrix} \beta \\ \alpha\,\alpha \end{matrix} \right\} = \frac{1}{2g} \left(-g_{\alpha\beta} \frac{\partial g_{\alpha\alpha}}{\partial u^{\alpha}} + 2\, g_{\alpha\alpha} \frac{\partial g_{\alpha\beta}}{\partial u^{\alpha}} - g_{\alpha\alpha} \frac{\partial g_{\alpha\alpha}}{\partial u^{\beta}} \right)$$

$$\left\{ \begin{matrix} \alpha \\ \alpha\,\beta \end{matrix} \right\} = \frac{1}{2g} \left(g_{\beta\beta} \frac{\partial g_{\alpha\beta}}{\partial u^{\beta}} - g_{\alpha\beta} \frac{\partial g_{\beta\beta}}{\partial u^{\alpha}} \right)$$

........................(2.42)

Didefinisikan pula bahwa :

$$g^{\beta\alpha}\, g_{\alpha\gamma} = \delta^{\beta}_{\gamma} \qquad \text{.........................} (2.43)$$

Maka :

$$g_{\omega\delta} \left\{ \begin{matrix} \delta \\ \alpha\,\beta \end{matrix} \right\} = g_{\omega\delta}\, g^{\delta\gamma} [\alpha\beta, \gamma]$$

$$= \delta^{\gamma}_{\omega} [\alpha\beta, \gamma] = [\alpha\beta, \gamma]$$

Sebutlah :

$$g_{\gamma\delta} \left\{ \begin{matrix} \delta \\ \alpha\,\beta \end{matrix} \right\} = [\alpha\beta, \gamma] \qquad \text{.......................} (2.44)$$

Dari (2.41) kita peroleh :

$$2 [\alpha\beta, \gamma] = \frac{\partial g_{\alpha\gamma}}{\partial u^{\beta}} + \frac{\partial g_{\beta\gamma}}{\partial u^{\alpha}} - \frac{\partial g_{\alpha\beta}}{\partial u^{\gamma}}$$

$$\frac{\partial g_{\alpha\gamma}}{\partial u^{\beta}} = 2 [\alpha\beta, \gamma] + \frac{\partial g_{\alpha\beta}}{\partial u^{\gamma}} - \frac{\partial g_{\beta\gamma}}{\partial u^{\beta}}$$

$$= 2 [\alpha\beta, \gamma] + \frac{\partial g_{\alpha\gamma}}{\partial u^{\beta}} + \frac{\partial g_{\alpha\beta}}{\partial u^{\gamma}} - \frac{\partial g_{\beta\gamma}}{\partial u^{\alpha}} + $$

$$+ \frac{\partial g_{\alpha\gamma}}{\partial u^{\beta}}$$

$$2 \frac{\partial g_{\alpha\gamma}}{\partial u^{\beta}} = 2 [\alpha\beta, \gamma] + \left(\frac{\partial g_{\alpha\gamma}}{\partial u^{\beta}} + \frac{\partial g_{\alpha\beta}}{\partial u^{\gamma}} - \frac{\partial g_{\beta\gamma}}{\partial u^{\alpha}} \right)$$

$$\frac{\partial g_{\alpha\gamma}}{\partial u^{\beta}} = [\alpha\beta, \gamma] + \tfrac{1}{2} \left(\frac{\partial g_{\alpha\gamma}}{\partial u^{\beta}} + \frac{\partial g_{\alpha\beta}}{\partial u^{\gamma}} - \frac{\partial g_{\beta\gamma}}{\partial u^{\alpha}} \right)$$

$$\frac{\partial g_{\alpha\gamma}}{\partial u^{\beta}} = [\alpha\beta, \gamma] + [\beta\gamma, \alpha] \qquad \text{..................} (2.45)$$

$$g_{\alpha\gamma}\ \frac{\partial g^{\beta\alpha}}{\partial u^{\theta}} + g^{\beta\alpha}\ \frac{\partial g_{\alpha\gamma}}{\partial u^{\theta}} = 0$$

Persamaan ini dikalikan g , maka :

$$g_{\alpha\gamma}\ g^{\gamma\delta}\ \frac{\partial g^{\beta\alpha}}{\partial u^{\theta}} + g^{\beta\alpha}\ g^{\gamma\delta}\ \frac{\partial g_{\alpha\gamma}}{\partial u^{\theta}} = 0$$

$$\delta^{\delta}_{\alpha}\ \frac{\partial g^{\beta\alpha}}{\partial u^{\theta}} + g^{\beta\alpha}\ g^{\gamma\delta}\ \frac{\partial g_{\alpha\gamma}}{\partial u^{\theta}} = 0$$

$$\frac{\partial g^{\beta\alpha}}{\partial u^{\theta}} = -\, g^{\beta\alpha}\ g^{\gamma\delta}\ \frac{\partial g_{\alpha\gamma}}{\partial u^{\theta}}$$

$$= -\, g^{\beta\alpha}\ g^{\gamma\delta}\ \left([\alpha\theta, \gamma] + [\theta\gamma, \alpha] \right)$$

$$= -\, g^{\beta\alpha}\ g^{\gamma\delta} [\alpha\theta, \gamma] - g^{\gamma\delta}\ g^{\beta\alpha} [\theta\gamma, \alpha]$$

$$\frac{\partial g^{\beta\alpha}}{\partial u^{\theta}} = -\, g^{\beta\alpha} \begin{Bmatrix} \delta \\ \alpha\theta \end{Bmatrix} - g^{\gamma\delta} \begin{Bmatrix} \beta \\ \theta\gamma \end{Bmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2.46)$$

Didefinisikan :

$$g = | g_{\alpha\beta} |$$

$$A = g^{\alpha\beta}\ g$$

$$\frac{\partial g}{\partial u^{\gamma}} = \frac{\partial g_{\alpha\beta}}{\partial u^{\gamma}}\ A = \frac{\partial g_{\alpha\beta}}{\partial u^{\gamma}}\ g^{\alpha\beta}\ g$$

Maka : $\dfrac{\partial g}{\partial u^{\gamma}} = g\ g^{\alpha\beta}\ \dfrac{\partial g_{\alpha\beta}}{\partial u^{\gamma}}$ (sesuai (2.44))

$$= g\ g^{\alpha\beta}\ \left([\alpha\gamma, \beta] + [\beta\gamma, \alpha] \right)$$

$$= g\ \left(g^{\alpha\beta} [\alpha\gamma, \beta] + g^{\alpha\beta} [\beta\gamma, \alpha] \right)$$

$$= g\ \left(g^{\beta\alpha} [\alpha\gamma, \beta] + g^{\alpha\beta} [\beta\gamma, \alpha] \right)$$

(sesuai (2.41))

Jadi : $\frac{\partial g}{\partial u^\gamma} = 2\ g \begin{Bmatrix} \alpha \\ \alpha\ \gamma \end{Bmatrix}$

$$\frac{1}{2g} \frac{\partial g}{\partial u^\gamma} = \begin{Bmatrix} \alpha \\ \alpha\ \gamma \end{Bmatrix}$$

$$\frac{\partial \log \sqrt{g}}{\partial u^\gamma} = \begin{Bmatrix} \alpha \\ \alpha\ \gamma \end{Bmatrix} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.47)$$

Dari 92.45), (2.46), (2.47) kita peroleh :

$$\frac{\partial g_{\alpha\beta}}{\partial u^\gamma} = g_{\beta\delta} \begin{Bmatrix} \delta \\ \alpha\gamma \end{Bmatrix} + g_{\alpha\delta}$$

$$\frac{\partial g^{\beta\alpha}}{\partial u^\gamma} = -g^{\beta\alpha} \begin{Bmatrix} \delta \\ \alpha\gamma \end{Bmatrix} - g^{\theta\delta} \begin{Bmatrix} \beta \\ \gamma\omega \end{Bmatrix} \ldots\ldots\ldots\ldots\ldots (2.48)$$

$$\frac{\partial \log \sqrt{g}}{\partial u^\alpha} = \begin{Bmatrix} \beta \\ \beta\alpha \end{Bmatrix}$$

Dalam simbol - simbol Christoffel berlaku pertukaran/pergan tian indeks dalam operasinya, sebagai contoh :

Akan kita tunjukkan bahwa : $\begin{Bmatrix} \beta \\ \alpha\alpha \end{Bmatrix} = \begin{Bmatrix} \alpha \\ \beta\beta \end{Bmatrix}$

$$[ij,k] = \tfrac{1}{2} \left(\frac{\partial g_{ik}}{\partial u^j} + \frac{\partial g_{jk}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^k} \right)$$

Maka :

$$[\beta\beta,\alpha] = \tfrac{1}{2} \left(\frac{\partial g_{\beta\alpha}}{\partial u^\beta} + \frac{\partial g_{\beta\alpha}}{\partial u^\beta} - \frac{\partial g_{\beta\beta}}{\partial u^\alpha} \right)$$

$$[\beta\beta,\beta] = \tfrac{1}{2} \left(\frac{\partial g_{\beta\beta}}{\partial u^\beta} + \frac{\partial g_{\beta\beta}}{\partial u^\beta} - \frac{\partial g_{\beta\beta}}{\partial u^\beta} \right)$$

Sehingga :

$$\begin{Bmatrix} \alpha \\ \beta\beta \end{Bmatrix} = g^{\alpha\alpha} [\beta\beta, \alpha] + g^{\alpha\beta} [\beta\beta, \beta]$$

$$= g^{\alpha\alpha}\ \tfrac{1}{2} \left(2 \frac{\partial g_{\alpha\beta}}{\partial u^\beta} - \frac{\partial g_{\beta\beta}}{\partial u^\alpha} \right) + \tfrac{1}{2} \left(\frac{\partial g_{\beta\beta}}{\partial u^\beta} \right) g^{\alpha\beta}$$

$$= \tfrac{1}{2} \left(g^{\alpha\alpha} \left(2 \frac{\partial g_{\alpha\beta}}{\partial u^\beta} \right) + g^{\alpha\beta} \frac{\partial g_{\beta\beta}}{\partial u^\beta} - g^{\alpha\alpha} \frac{\partial g_{\beta\beta}}{\partial u^\alpha} \right)$$

Dengan menggantikan indeks kita mendapatkan :

$$\left\{ {\beta \atop \alpha\,\alpha} \right\} = \frac{1}{2g} \left(- g_{\alpha\beta} \frac{\partial g_{\beta\beta}}{\partial u^{\beta}} + 2\, g_{\beta\beta} \frac{\partial g_{\alpha\beta}}{\partial u^{\beta}} - g_{\alpha\alpha} \frac{\partial g_{\beta\beta}}{\partial u^{\alpha}} \right)$$

$$\left\{ {\beta \atop \alpha\alpha} \right\} = \frac{1}{2g} \left(- g_{\alpha\beta} \frac{\partial g_{\beta\beta}}{\partial u^{\beta}} + 2\, g_{\beta\beta} \frac{\partial g_{\alpha\beta}}{\partial u^{\beta}} - g_{\alpha\alpha} \frac{\partial g_{\beta\beta}}{\partial u^{\alpha}} \right)$$

Terlihatlah bahwa : $\left\{ {\alpha \atop \beta\beta} \right\} = \left\{ {\beta \atop \alpha\alpha} \right\}$

Sekarang kita bicarakan hubungan antara simbol - simbol Christoffel jenis ke dua dalam dua sistem koordinat.

Diberikan persamaan :

$$g'_{pq} = g_{ij} \frac{\partial u^i}{\partial u'^p} \frac{\partial u^j}{\partial u'^q} \quad \ldots\ldots\ldots\ldots\ldots \quad (2.49)$$

$$(\quad i,j,p,q = 1,2,3, \ldots\ldots \quad)$$

Did ferensialkan ke u'^r, maka :

$$\frac{\partial g'_{pq}}{\partial u'^r} = \frac{\partial g_{ij}}{\partial u^k} \frac{\partial u^i}{\partial u'^p} \frac{\partial u^j}{\partial u'^q} \frac{\partial u^k}{\partial u'^r} + g_{ij} \left(\frac{\partial^2 u^i}{\partial u'^p \partial u'^r} \frac{\partial u^j}{\partial u'^q} + \frac{\partial u^j}{\partial u'^p} \frac{\partial^2 u^i}{\partial u'^q \partial u'^r} \right)$$

dengan mengadakan penukaran indeks bebas dan pengganti, maka :

$$\frac{\partial g'_{rq}}{\partial u'^p} = \frac{\partial g_{kj}}{\partial u^i} \frac{\partial u^i}{\partial u'^p} \frac{\partial u^j}{\partial u'^q} \frac{\partial u^k}{\partial u'^r} + g_{ij} \left(\frac{\partial^2 u^i}{\partial u'^p \partial u'^q} \frac{\partial u^j}{\partial u'^r} + \frac{\partial^2 u^j}{\partial u'^q \partial u'^r} \frac{\partial u^i}{\partial u'^p} \right)$$

$$\frac{\partial g'_{pr}}{\partial u'^q} = \frac{\partial g_{ik}}{\partial u^j} \frac{\partial u^i}{\partial u'^p} \frac{\partial u^j}{\partial u'^q} \frac{\partial u^k}{\partial u'^r} + g_{ij} \left(\frac{\partial^2 u^i}{\partial u'^p \partial u'^q} \frac{\partial u^j}{\partial u'^r} + \frac{\partial^2 u^j}{\partial u'^q \partial u'^r} \frac{\partial u^i}{\partial u'^p} \right)$$

(ditambahkan)

$$\frac{\partial g'_{rq}}{\partial u'^p} + \frac{\partial g'_{pr}}{\partial u'^q} = \left(\frac{\partial g_{kj}}{\partial u^i} + \frac{\partial g_{ik}}{\partial u^j} \right) \frac{\partial u^i}{\partial u'^p} \frac{\partial u^j}{\partial u'^q} \frac{\partial u^k}{\partial u'^r} +$$

$$+ 2\, g_{ij} \left(\frac{\partial^2 u^i}{\partial u'^p \partial u'^q} \frac{\partial u^j}{\partial u'^r} + \frac{\partial^2 u^j}{\partial u'^q \partial u'^r} \frac{\partial u^i}{\partial u'^p} \right)$$

hasil ini dikurangi dengan :

$$\frac{\partial g'_{pq}}{\partial u'^r} = \frac{\partial g_{ij}}{\partial u^k} \frac{\partial u^i}{\partial u'^p} \frac{\partial u^j}{\partial u'^q} \frac{\partial u^k}{\partial u'^r} + g_{ij} \left(\frac{\partial^2 u^i}{\partial u'^p \partial u'^q} \frac{\partial u^j}{\partial u'^r} + \frac{\partial^2 u^j}{\partial u'^q \partial u'^r} \frac{\partial u^i}{\partial u'^p} \right)$$

Maka hasil akhirnya adalah :

$$\frac{\partial g'_{rq}}{\partial u'^p} + \frac{\partial g'_{pr}}{\partial u'^q} - \frac{\partial g'_{pq}}{\partial u'^r} = \left(\frac{\partial g_{kj}}{\partial u^i} + \frac{\partial g_{ik}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^k} \right) \frac{\partial u^i}{\partial u'^p} \frac{\partial u^j}{\partial u'^q} \frac{\partial u^k}{\partial u'^r} +$$

$$+ g_{ij} \left(\frac{\partial^2 u^i}{\partial u'^p \partial u'^r} \frac{\partial u^j}{\partial u'^q} + \frac{\partial^2 u^j}{\partial u'^q \partial u'^r} \frac{\partial u^i}{\partial u'^p} \right)$$

$$\frac{\partial g'_{rq}}{\partial u'^p} + \frac{\partial g'_{pr}}{\partial u'^q} - \frac{\partial g'_{pq}}{\partial u'^r} = \left(\frac{\partial g_{kj}}{\partial u^i} + \frac{\partial g_{ik}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^k} \right) \frac{\partial u^i}{\partial u'^p} \frac{\partial u^j}{\partial u'^q} \frac{\partial u^k}{\partial u'^r} +$$

$$+ g_{ij} \left(\frac{\partial u^i}{\partial u'^r} \frac{\partial^2 u^j}{\partial u'^p\, u'^q} + \frac{\partial u^i}{\partial u'^r} \frac{\partial^2 u^j}{\partial u'^p \partial u'^q} \right)$$

$$= \left(\frac{\partial g_{kj}}{\partial u^i} + \frac{\partial g_{ik}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^k} \right) \frac{\partial u^i}{\partial u'^p} \frac{\partial u^j}{\partial u'^q} \frac{\partial u^k}{\partial u'^r} +$$

$$+ 2\, g_{ij} \frac{\partial u^i}{\partial u'^r} \frac{\partial^2 u^j}{\partial u'^p \partial u'^q}$$

$$\tfrac{1}{2} \left(\frac{\partial g'_{rq}}{\partial u'^p} + \frac{\partial g'_{pr}}{\partial u'^q} - \frac{\partial g'_{pq}}{\partial u'^r} \right) =$$

$$= \tfrac{1}{2} \left(\frac{\partial g_{kj}}{\partial u^i} + \frac{\partial g_{ik}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^k} \right) \frac{\partial u^i}{\partial u'^p} \frac{\partial u^j}{\partial u'^q} \frac{\partial u^k}{\partial u'^r} + g_{ij} \left(\frac{\partial u^i}{\partial u'^r} \frac{\partial^2 u^j}{\partial u'^p \partial u'^q} \right)$$

$$[pq,r]' = [ij,k] \frac{\partial g_{kj}}{\partial u^i} \frac{\partial g_{ik}}{\partial u^j} \frac{\partial g_{ij}}{\partial u^k} + g_{ij} \frac{\partial u^i}{\partial u'^r} \frac{\partial^2 u^j}{\partial u'^p \partial u'^q} \qquad \text{.............................. (2.50)}$$

Mengingat bentuk :

$$g'^{sr} = g^{hl} \frac{\partial u'^s}{\partial u^h} \frac{\partial u'^r}{\partial u^l} \quad \text{atau;}$$

$$g'^{sr} \frac{\partial u^h}{\partial u'^s} = g^{hl} \frac{\partial u'^r}{\partial u^l}$$

Ruas kiri persamaan (2.50) dikalikan $g'^{sr} \dfrac{\partial u^h}{\partial u'^s}$ dan

ruas kanannya dikalikan $g^{hl} \dfrac{\partial u'^r}{\partial u^l}$; maka :

$$g'^{sr} [pq,r]' \frac{\partial u^h}{\partial u'^s} = [ij,k] \frac{\partial u^i}{\partial u'^p} \frac{\partial u^j}{\partial u'^q} g^{hl} \frac{\partial u'^r}{\partial u^l} \frac{\partial u^k}{\partial u'^r} +$$

dan kemudian jika kita menggunakan persamaan (2.41) dengan penyesuaian indeks - indeks yang sesuai, akan kita peroleh:

$$\left\{ \begin{matrix} s \\ p\ q \end{matrix} \right\}' \frac{\partial u^h}{\partial u'^s} = [ij,k] \frac{\partial u^i}{\partial u'^p} \frac{\partial u^j}{\partial u'^q} g^{hl} \delta^k_l + g_{ij} g^{hl} \delta^i_l \frac{\partial^2 u^j}{\partial u'^p \partial u'^q}$$

$$= [ij,k] \frac{\partial u^i}{\partial u'^p} \frac{\partial u^j}{\partial u'^q} g^{hk} + g_{ij} g^{hi} \frac{\partial^2 u^j}{\partial u'^p \partial u'^q}$$

$$= g^{hk} [ij,k] \frac{\partial u^i}{\partial u'^p} \frac{\partial u^j}{\partial u'^q} + \delta^h_j \frac{\partial^2 u^j}{\partial u'^p \partial u'^q}$$

$$\left\{ \begin{matrix} s \\ p\ q \end{matrix} \right\}' \frac{\partial u^h}{\partial u'^s} = \frac{\partial^2 u^j}{\partial u'^p \partial u'^q} + \left\{ \begin{matrix} h \\ i\ j \end{matrix} \right\} \frac{u^i}{\partial u'^p} \frac{u^j}{\partial u'^q} \qquad \text{.............................. (2.51)}$$

Simbol - simbol <u>RIEMAN</u> diturunkan dari simbol - simbol - Christoffel.

Persamaan (2.51) didiferensialkan ke u'^r, maka :

$$\frac{\partial}{\partial u'^r} \frac{\partial^2 u^j}{\partial u'^p \partial u'^q} + \frac{\partial}{\partial u'^r} \left\{ \begin{matrix} h \\ i\ j \end{matrix} \right\} \frac{\partial u^i}{\partial u'^p} \frac{\partial u^j}{\partial u'^q} +$$

$$+ \left\{ \begin{matrix} h \\ i\ j \end{matrix} \right\} \left(\frac{\partial^2 u^i}{\partial u'^p \partial u'^r} \frac{\partial u^j}{\partial u'^q} + \frac{\partial u^i}{\partial u'^p} \frac{\partial^2 u^j}{\partial u'^q \partial u'^r} \right) =$$

$$= \frac{\partial}{u'^r} \left\{ \begin{matrix} s \\ p\ q \end{matrix} \right\}' \frac{\partial u^h}{\partial u'^s} + \left\{ \begin{matrix} s \\ p\ q \end{matrix} \right\}' \frac{\partial^2 u^h}{\partial u'^s \partial u'^r}$$

diadakan penukaran indeks q dan r , maka :

$$\frac{\partial}{\partial u'^q} \frac{\partial^2 u^j}{\partial u'^p \partial u'^r} + \frac{\partial}{\partial u'^q} \left\{ \begin{matrix} h \\ i\ j \end{matrix} \right\} \frac{\partial u^i}{\partial u'^p} \frac{\partial u^j}{\partial u'^r} +$$

$$+ \left\{ \begin{matrix} h \\ i\ j \end{matrix} \right\} \left(\frac{\partial^2 u^i}{\partial u'^p \partial u'^q} \frac{\partial u^j}{\partial u'^r} + \frac{\partial u^i}{\partial u'^p} \frac{\partial^2 u^j}{\partial u'^q \partial u'^r} \right) =$$

$$= \frac{\partial}{\partial u'^q} \left\{ \begin{matrix} s \\ p\ q \end{matrix} \right\}' \frac{\partial u^h}{\partial u'^s} + \left\{ \begin{matrix} s \\ p\ q \end{matrix} \right\}' \frac{\partial^2 u^j}{\partial u'^q \partial u'^s}$$

Dengan mengubah cara penulisan sehingga kedua persamaan tersebut akanmenjadi lebih sederhana ;

$$\frac{\partial^3 u^j}{\partial u'^p \partial u'^q \partial u'^r} + \frac{\partial}{\partial u'^k}\left\{\begin{matrix} h \\ i\ j \end{matrix}\right\} \frac{\partial u^i}{\partial u'^p}\ \frac{\partial u^j}{\partial u'^q}\ \frac{\partial u^k}{\partial u'^r} +$$

$$+ \left\{\begin{matrix} h \\ i\ j \end{matrix}\right\} \left(\frac{\partial^2 u^i}{\partial u'^p\, \partial u'^r}\ \frac{\partial u^j}{\partial u'^q} + \frac{\partial u^i}{\partial u'^p}\ \frac{\partial^2 u^j}{\partial u'^q\, \partial u'^r} \right) =$$

$$= \frac{\partial}{\partial u'^r}\left\{\begin{matrix} s \\ p\ q \end{matrix}\right\}' \frac{\partial u^h}{\partial u'^s} + \left\{\begin{matrix} s \\ p\ q \end{matrix}\right\}' \frac{\partial^2 u^h}{\partial u'^s\, \partial u'^r}$$

$$\frac{\partial^3 u^j}{\partial u'^p \partial u'^q \partial u'^r} + \frac{\partial}{\partial u'^j}\left\{\begin{matrix} h \\ i\ k \end{matrix}\right\} \frac{\partial u^i}{\partial u'^p}\ \frac{\partial u^j}{\partial u'^q}\ \frac{\partial u^k}{\partial u'^r} +$$

$$+ \left\{\begin{matrix} h \\ i\ j \end{matrix}\right\} \left(\frac{\partial^2 u^i}{\partial u'^p \partial u'^q}\ \frac{\partial u^j}{\partial u'^r} + \frac{\partial u^i}{\partial u'^p}\ \frac{\partial^2 u^j}{\partial u'^q \partial u'^r} \right) =$$

$$= \frac{\partial}{\partial u'^q}\left\{\begin{matrix} s \\ p\ r \end{matrix}\right\}' \frac{\partial u^h}{\partial u'^s} + \left\{\begin{matrix} s \\ p\ r \end{matrix}\right\}' \frac{\partial^2 u^h}{\partial u'^s\, \partial u'^q}$$

Kedua persamaan tersebut dikurangkan, sehingga kita peroleh hasil sebagai berikut :

$$\left(\frac{\partial}{\partial u'^k}\left\{\begin{matrix} h \\ i\ j \end{matrix}\right\} - \frac{\partial}{\partial u'^j}\left\{\begin{matrix} h \\ i\ k \end{matrix}\right\} \right) \frac{\partial u^i}{\partial u'^p}\ \frac{\partial u^j}{\partial u'^q}\ \frac{\partial u^k}{\partial u'^r} +$$

$$+ \left\{\begin{matrix} h \\ i\ j \end{matrix}\right\} \left(\frac{\partial^2 u^i}{\partial u'^p\, \partial u'^r}\ \frac{\partial u^j}{\partial u'^q} - \frac{\partial u^j}{\partial u'^r}\ \frac{\partial^2 u^i}{\partial u'^p\, \partial u'^q} \right) =$$

$$= \left(\frac{\partial}{\partial u'^r}\left\{\begin{matrix} s \\ p\ q \end{matrix}\right\}' - \frac{\partial}{\partial u'^r}\left\{\begin{matrix} s \\ p\ r \end{matrix}\right\}' \right) \frac{\partial u^h}{\partial u'^s} +$$

$$+ \left\{\begin{matrix} s \\ p\ q \end{matrix}\right\}' \frac{\partial^2 u^h}{\partial u'^s \partial u'^r} - \left\{\begin{matrix} s \\ p\ r \end{matrix}\right\}' \frac{\partial^2 u^h}{\partial u'^s\, \partial u'^q}$$

.... (2.52)

Dengan menggunakan bentuk persamaan (2.51) terdapat bentuk :

$$\frac{\partial^2 u^i}{\partial u'^p \, \partial u'^q} = \begin{Bmatrix} s \\ p \; q \end{Bmatrix}' \frac{\partial u^i}{\partial u'^s} - \begin{Bmatrix} h \\ i \; j \end{Bmatrix} \frac{\partial u^i}{\partial u'^p} \frac{\partial u^j}{\partial u'^q}$$

$$\frac{\partial^2 u^i}{\partial u'^p \, \partial u'^r} = \begin{Bmatrix} s \\ p \; r \end{Bmatrix}' \frac{\partial u^i}{\partial u'^s} - \begin{Bmatrix} h \\ i \; j \end{Bmatrix} \frac{\partial u^i}{\partial u'^p} \frac{\partial u^j}{\partial u'^r}$$

$$\frac{\partial^2 u^h}{\partial u'^s \, \partial u'^r} = \begin{Bmatrix} s \\ s \; r \end{Bmatrix}' \frac{\partial u^h}{\partial u'^s} - \begin{Bmatrix} i \\ h \; j \end{Bmatrix} \frac{\partial u^h}{\partial u'^s} \frac{\partial u^j}{\partial u'^r}$$

$$\frac{\partial^2 u^h}{\partial u'^s \, \partial u'^q} = \begin{Bmatrix} s \\ s \; q \end{Bmatrix}' \frac{\partial u^h}{\partial u'^s} - \begin{Bmatrix} i \\ h \; j \end{Bmatrix} \frac{\partial u^h}{\partial u'^s} \frac{\partial u^j}{\partial u'^q}$$

Persamaan - persamaan ini disubstitusikan ke persamaan (2.51), maka kita peroleh bentuk persamaan :

$$R^h_{ijk} \frac{\partial u^i}{\partial u'^p} \frac{\partial u^j}{\partial u'^q} \frac{\partial u^k}{\partial u'^r} = R'^s_{pqr} \frac{\partial u^h}{\partial u'^s} \quad \ldots\ldots\ldots (2.53)$$

dengan R^h_{ijk} didefinisikan sebagai :

$$R^h_{ijk} = \frac{\partial}{\partial u^j} \begin{Bmatrix} h \\ i \; k \end{Bmatrix} - \frac{\partial}{\partial u^k} \begin{Bmatrix} h \\ i \; j \end{Bmatrix} + \begin{Bmatrix} h \\ i \; j \end{Bmatrix}\begin{Bmatrix} p \\ l \; k \end{Bmatrix} - \begin{Bmatrix} h \\ i \; k \end{Bmatrix}\begin{Bmatrix} p \\ l \; j \end{Bmatrix}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.54)$$

dengan R'^s_{pqr} mempunyai bentuk persamaan yang sama.

Persamaan (2.53) dikalikan $\frac{\partial u'^t}{\partial u^h}$, maka :

$$R^h_{ijk} \frac{\partial u^i}{\partial u'^p} \frac{\partial u^j}{\partial u'^q} \frac{\partial u^k}{\partial u'^r} \frac{\partial u'^t}{\partial u^h} = R'^s_{pqr} \frac{\partial u^h}{\partial u'^s} \frac{\partial u'^t}{\partial u^h}$$

$$R^h_{ijk} \frac{\partial u^i}{\partial u'^p} \frac{\partial u^j}{\partial u'^q} \frac{\partial u^k}{\partial u'^r} \frac{\partial u'^t}{\partial u^h} = R'^t_{pqr} \frac{\partial u^h}{\partial u'^t} \frac{\partial u'^t}{\partial u^h}$$

$$R^h_{ijk} \frac{\partial u^i}{\partial u'^p} \frac{\partial u^j}{\partial u'^q} \frac{\partial u^k}{\partial u'^r} \frac{\partial u'^t}{\partial u^h} = R'^t_{pqr}$$

R^{h}_{ijk} disebut sebagai Simbol Rieman Jenis Ke Dua, juga merupakan komponen - komponen sebuah tensor kontravarian orde pertama dan tensor kovarian orde ke tiga.

Telah didefinisikan bentuk pertama tensor Rieman, yaitu :

$$R_{\alpha\beta\gamma\delta} = g_{\alpha\theta} R^{\theta}_{\beta\gamma\delta}$$

Maka dari persamaan (2.54) kita peroleh :

$$R_{\alpha\beta\gamma\delta} = g_{\alpha\theta} \left(\frac{\partial}{\partial u^{\gamma}} \begin{Bmatrix} \theta \\ \beta\delta \end{Bmatrix} - \frac{\partial}{\partial u^{\delta}} \begin{Bmatrix} \theta \\ \beta\gamma \end{Bmatrix} + \begin{Bmatrix} \tau \\ \beta\gamma \end{Bmatrix} \begin{Bmatrix} \theta \\ \delta\alpha \end{Bmatrix} - \begin{Bmatrix} \tau \\ \beta\delta \end{Bmatrix} \begin{Bmatrix} \theta \\ \gamma\alpha \end{Bmatrix} \right) \quad \text{.........................(2.55)}$$

(tentu saja sudah dengan penyesuaian indeks)

$$R_{\alpha\beta\gamma\delta} = g_{\alpha\theta} \frac{\partial}{\partial u^{\gamma}} \left(g^{\varepsilon\theta} [\beta\delta,\varepsilon] \right) - g_{\alpha\theta} \frac{\partial}{\partial u^{\delta}} \left(g^{\varepsilon\theta} [\beta\gamma,\varepsilon] \right) +$$

$$+ \; g^{\alpha\tau} \left([\beta\gamma,\alpha] \begin{Bmatrix} \theta \\ \delta\alpha \end{Bmatrix} - [\beta\delta,\alpha] \begin{Bmatrix} \theta \\ \gamma\alpha \end{Bmatrix} \right)$$

$$= g_{\alpha\theta}\, g^{\varepsilon\theta} \frac{\partial}{\partial u^{\gamma}} [\beta\delta,\varepsilon] - g_{\alpha\theta}\, g^{\varepsilon\theta} \frac{\partial}{\partial u^{\delta}} [\beta\gamma,\varepsilon] +$$

$$+ \; g_{\alpha\theta}\, g^{\alpha\tau} \left([\beta\gamma,\alpha] \begin{Bmatrix} \theta \\ \delta\alpha \end{Bmatrix} - [\beta\delta,\alpha] \begin{Bmatrix} \theta \\ \gamma\alpha \end{Bmatrix} \right)$$

$$= \delta^{\varepsilon}_{\alpha} \frac{\partial}{\partial u^{\gamma}} [\beta\delta,\varepsilon] - \delta^{\varepsilon}_{\alpha} \frac{\partial}{\partial u^{\delta}} [\beta\gamma,\varepsilon] +$$

$$+ \; \delta^{\tau}_{\theta} \left([\beta\gamma,\alpha] \begin{Bmatrix} \theta \\ \delta\alpha \end{Bmatrix} - [\beta\delta,\alpha] \begin{Bmatrix} \theta \\ \gamma\alpha \end{Bmatrix} \right)$$

$$= \frac{\partial}{\partial u^{\gamma}} [\beta\delta,\varepsilon] - \frac{\partial}{\partial u^{\delta}} [\beta\gamma,\varepsilon] + [\beta\gamma,\alpha] \begin{Bmatrix} \theta \\ \delta\alpha \end{Bmatrix} - [\beta\delta,\alpha] \begin{Bmatrix} \theta \\ \gamma\alpha \end{Bmatrix}$$

$$= \frac{\partial}{\partial u^{\gamma}} \left([\beta\delta,\varepsilon] - \frac{\partial}{\partial u^{\delta}} [\beta\delta,\varepsilon] + g^{\omega\theta} \left(\begin{Bmatrix} \omega \\ \beta\gamma \end{Bmatrix} \begin{Bmatrix} \theta \\ \delta\alpha \end{Bmatrix} - \begin{Bmatrix} \omega \\ \beta\delta \end{Bmatrix} \begin{Bmatrix} \theta \\ \gamma\alpha \end{Bmatrix} \right) \right.$$

Maka dengan mensubstitusikan persamaan (2.41) terdapat :

$$R_{\alpha\beta\gamma\delta} = \frac{\partial}{\partial u^{\gamma}} \tfrac{1}{2} \left(\frac{\partial g_{\delta\alpha}}{\partial u^{\beta}} + \frac{\partial g_{\beta\alpha}}{\partial u^{\delta}} - \frac{\partial g_{\beta\delta}}{\partial u^{\alpha}} \right) +$$

$$- \frac{\partial}{\partial u^{\delta}} \tfrac{1}{2} \left(\frac{\partial g_{\gamma\alpha}}{\partial u^{\beta}} + \frac{\partial g_{\beta\alpha}}{\partial u^{\gamma}} - \frac{\partial g_{\beta\gamma}}{\partial u^{\alpha}} \right) +$$

$$+ g^{\omega\theta} \left(\left\{ \begin{matrix} \omega \\ \beta\gamma \end{matrix} \right\} \left\{ \begin{matrix} \theta \\ \delta\alpha \end{matrix} \right\} - \left\{ \begin{matrix} \omega \\ \beta\delta \end{matrix} \right\} \left\{ \begin{matrix} \theta \\ \gamma\alpha \end{matrix} \right\} \right)$$

$$= \tfrac{1}{2} \left(\frac{\partial^2 g_{\delta\alpha}}{\partial u^{\gamma}\, \partial u^{\beta}} + \frac{\partial^2 g_{\beta\alpha}}{\partial u^{\gamma}\, \partial u^{\delta}} - \frac{\partial^2 g_{\beta\delta}}{\partial u^{\alpha}\, \partial u^{\gamma}} - \frac{\partial^2 g_{\gamma\alpha}}{\partial u^{\beta} \partial u^{\delta}} \right) +$$

$$- \frac{\partial^2 g_{\beta\alpha}}{\partial u^{\gamma}\, \partial u^{\delta}} \quad \frac{\partial^2 g_{\beta\alpha}}{\partial u^{\alpha}\, \partial u^{\delta}} \Bigg) + g^{\omega\theta} \left(\left\{ \begin{matrix} \omega \\ \beta\gamma \end{matrix} \right\} \left\{ \begin{matrix} \theta \\ \delta\alpha \end{matrix} \right\} - \left\{ \begin{matrix} \omega \\ \beta\delta \end{matrix} \right\} \left\{ \begin{matrix} \theta \\ \gamma\alpha \end{matrix} \right\} \right)$$

$$R_{\alpha\beta\gamma\delta} = \tfrac{1}{2} \left(\frac{\partial^2 g_{\delta\alpha}}{\partial u^{\gamma} \partial u^{\beta}} + \frac{\partial^2 g_{\beta\alpha}}{\partial u^{\alpha}\, \partial u^{\delta}} - \frac{\partial^2 g_{\beta\delta}}{\partial u^{\alpha} \partial u^{\gamma}} - \frac{\partial^2 g_{\gamma\alpha}}{\partial u^{\beta}\, \partial u^{\delta}} \right) +$$

$$+ g^{\omega\theta} \left(\left\{ \begin{matrix} \omega \\ \beta\gamma \end{matrix} \right\} \left\{ \begin{matrix} \theta \\ \delta\alpha \end{matrix} \right\} - \left\{ \begin{matrix} \omega \\ \beta\delta \end{matrix} \right\} \left\{ \begin{matrix} \theta \\ \gamma\alpha \end{matrix} \right\} \right)$$

.... (2.56)

$$R_{1212} = \tfrac{1}{2} \left(\frac{\partial^2 g_{12}}{\partial u^1\, \partial u^2} + \frac{\partial^2 g_{12}}{\partial u^2\, \partial u^1} - \frac{\partial^2 g_{22}}{\partial u^1\, \partial u^1} - \frac{\partial^2 g_{11}}{\partial u^2\, \partial u^2} \right) +$$

$$+ g^{\alpha\beta} \left(\left\{ \begin{matrix} \alpha \\ 1\ 2 \end{matrix} \right\} \left\{ \begin{matrix} \beta \\ 1\ 2 \end{matrix} \right\} - \left\{ \begin{matrix} \alpha \\ 2\ 2 \end{matrix} \right\} \left\{ \begin{matrix} \beta \\ 1\ 1 \end{matrix} \right\} \right)$$

$$R_{1212} = \tfrac{1}{2} \left(2 \frac{\partial^2 g_{12}}{\partial u^1\, \partial u^2} - \frac{\partial^2 g_{22}}{(\partial u^1)^2} - \frac{\partial^2 g_{11}}{(\partial u^2)^2} \right) + g^{\alpha\beta} \left(\left\{ \begin{matrix} \alpha \\ 1\ 2 \end{matrix} \right\} \left\{ \begin{matrix} \beta \\ 1\ 2 \end{matrix} \right\} - \left\{ \begin{matrix} \alpha \\ 2\ 2 \end{matrix} \right\} \left\{ \begin{matrix} \beta \\ 1\ 1 \end{matrix} \right\} \right)$$

.... (2.57)

Dari persamaan (2.56) diperoleh pula :

$R_{1212} = R_{2121} = - R_{2112} = - R_{1221}$

$R_{\alpha\alpha\beta\gamma} = R_{\alpha\beta\gamma\gamma} = 0$

Suatu permukaan pada bidang adalah isometri bila dan hanya bila Tensor Rieman adalah Tensor Nol.
Sehingga dapat kita katakan bahwa Tensor Rieman adalah besaran skalar (dalam hal ini kita pandang besarnya Tensor Rieman Bentuk Pertama yang telah dihitung seperti di atas dan kemudian dapat dipergunakan sebagai pedoman pembicaraan masalah di bawah ini).

Didefinisikan besarnya K, yaiyu : $K = \dfrac{R_{1212}}{g}$

disebut Lengkungan Gauss dari suatu permukaan dan juga merupakan Lengkungan Total pada suatu permukaan.

Definisi :

Lengkungan Gauss dari suatu permukaan adalah besaran skalar.
(sebab telah kita ketahui bahwa R_{1212} adalah besaran skalar)

Karena Tensor $g_{\alpha\beta}$ adalah Tensor Nol dan merupakan besaran skalar, maka dapat diperoleh :

Definisi :

Derivatip - derivatip kovarian dari $g_{\alpha\beta}$ dan $g^{\alpha\beta}$ adalah besaran skalar.

Definisi :

Derivatip - derivatip kovarian dari $\varepsilon_{\alpha\beta}$ dan $\varepsilon^{\alpha\beta}$ adalah besaran skalar.