BAB.II
FUNGSI BESSEL

Fungsi Bessel merupakan penyelesaian dari Persamaan Bessel, yaitu Persamaan Deferensial order dua yang mempunyai sebuah titik singuler reguler dan sebuah titik singuler irreguler.

Suatu fungsi f(x) dikatakan analitik pada suatu titik x = a, apabila fungsi tersebut dapat diperderetkan dalam bentuk deret Taylor disekitar titik x = a.

Dari suatu persamaan deferensial order dua :

$$y'' + \alpha(x)y' + \beta(x)y = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Apabila fungsi-fungsi $\alpha(x)$ dan $\beta(x)$, kedua-duanya atau salah satu tidak analitik dititik x = a, maka titik tersebut disebut titik singuler dari persamaan deferensial (1). Apabila $\alpha(x)$ dan $\beta(x)$ dapat ditulis sebagai :

$$\alpha(x) = \frac{P(x)}{x-a} \quad \text{dan} \quad \beta(x) = \frac{Q(x)}{(x-a)^2}$$

Sehingga persamaan (1) menjadi :

$$(x-a)^2 y'' + (x-a)P(x)y' + Q(x)y = 0$$

dimana fungsi-fungsi P(x) dan Q(x) analitik dititik x = a maka titik tersebut (x = a) disebut titik singuler reguler dari persamaan diferensial (1).

Apabila P(x) dan Q(x) salah satu atau kedua-duanya tidak analitik di titik x = a, maka titik tersebut disebut ti-

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tik singuler irreguler dari persamaan diferensial (1).

Bentuk kononik dari fungsi Bessel adalah :

$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + (x^2 - n^2)y = 0 \;.....\quad \text{II-1.}$$

dengan kedua titik singulernya yaitu

$x = 0$ merupakan titik singuler reguler

$x = \infty$ merupakan titik singuler irreguler

II.1. Mencari penyelesaian disekitar titik singuler reguler $x = 0$.

Penyelesaian disekitar titik tersebut adalah

$$y = \sum_{r=0}^{\infty} c_r \, x^{r+\rho}$$

$$y' = \sum_{r=0}^{\infty} (r+\rho)c_r \, x^{r+\rho-1}$$

$$y'' = \sum_{r=0}^{\infty} (r+\rho)(r+\rho-1)c_r \, x^{r+\rho-2}$$

Bila persamaan-persamaan ini disubstitusikan kepersamaan II-1 diperoleh :

$$\sum_{r=0}^{\infty} (r+\rho)(r+\rho-1)c_r \, x^{r+\rho} +$$

$$\sum_{r=0}^{\infty} (r+\rho)c_r \, x^{r+\rho} + (x^2-n^2)\sum_{r=0}^{\infty} c_r \, x^{r+\rho} = 0$$

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atau dapat ditulis :

$$\sum_{r=0}^{\infty} \left[\left\{ (r + \rho)^2 - n^2 \right\} C_r + C_{r-2} \right] x^{r+\rho} = 0$$

sehingga diperoleh rumus Rekurensi

$$\left\{ (r + \rho)^2 - n^2 \right\} C_r + C_{r-2} = 0 \qquad \text{II-2.}$$

dan rumus indicial

$$\rho^2 - n^2 = 0 \qquad \text{II-3.}$$

berarti : $\rho_{1.2} = \pm n$

Berdasarkan harga n yang diberikan, timbul beberapa kemungkinan :

a. $\rho_1 \neq \rho_2$ dan $\left| \rho_1 - \rho_2 \right| = 2n$ bukan bilangan bulat ($\neq 0$) yaitu bila n bukan bilangan bulat

b. $\rho_1 = \rho_2$, yaitu bila n = 0

c. $\rho_1 \neq \rho_2$ dan $\left| \rho_1 - \rho_2 \right| = 2n$ merupakan bilangan bulat ($\neq 0$) yaitu bila n bilangan bulat.

II.1.a. Ditinjau kemungkinan $\rho_1 \neq \rho_2$ dan $\left| \rho_1 - \rho_2 \right| = 2n$ bukan bilangan bulat.

Telah diketahui rumus rekurensi II - 2 berbentuk

$$\left\{ (r + \rho)^2 - n^2 \right\} C_r + C_{r-2} = 0$$

atau

$$C_r = \frac{-1}{(r+\rho+n)(r+\rho-n)} C_{r-2}$$

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untuk $\rho_1 = n$, maka didapat :

$$C_r = \frac{-1}{(r + 2n)\ (r)} C_{r-2}$$

Sehingga :

$$C_0 = \frac{-1}{0} C_{-2} = \frac{0}{0} = C \text{ (sebarang)}$$

$$C_1 = \frac{-1}{2n + 1} C_{-1} = 0 \quad \text{demikian pula untuk}$$

$$C_3\ ,\ C_5\ ,\ C_7\ ,\ldots$$

$$C_2 = \frac{-1}{2(2n + 2)} C_0 = \frac{-1}{4(n + 1)} C_0$$

$$C_4 = \frac{-1}{4(2n + 4)} C_2 = \frac{(-1)^2}{2^4\ .2(n + 2)(n + 1)} C_0$$

atau dapat ditulis :

$$C_{2r} = \frac{(-1)^r}{2^{2r}\ .\ r!(n+1)(n+2)\ldots\ldots(n+r)} C_0$$

$$= \frac{(-1)^r}{2^{2r}.r!} \frac{\Gamma(n + 1)}{\Gamma(n + r + 1)} C_0 \quad \ldots\ldots\ldots\ldots \text{II-4}$$

Sehingga diperoleh penyelesaian pertama yang berbentuk:

$$y_1 = \sum_{r=0}^{\infty} \frac{(-1)^r\ \Gamma(n + 1)}{2^{2r}.r!\,\Gamma(n+r+1)} C_0\ x^{2r+n}$$

Penyelesaian dasar Pertama didapat dengan mengambil :

$$C_0 = \frac{1}{2^n\ \Gamma(n+1)}$$

sehingga didapat

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n + r + 1)} \left(\frac{x}{2}\right)^{2r+n} \ldots\ldots \text{II-5.}$$

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Persamaan ini disebut fungsi Bessel macam pertama order n

Sedangkan untuk ρ_2 = -n didapat :

$$J_{-n}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(r-n+1)} \left(\frac{x}{2}\right)^{2r-n} \quad \ldots\ldots\ldots \text{ II-6}$$

Persamaan ini juga disebut fungsi Bessel macam kedua order n, yang independent terhadap $J_n(x)$ bila 2n bukan bilangan bulat.

Sehingga penyelesaian umum dari persamaan diferensial II-1 adalah :

$$y = C_1\, J_n(x) + C_2\, J_n(x) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{ II-7}$$

dengan C_1 dan C_2 konstanta sebarang

II.1.b. Ditinjau kemungkinan $\rho_1 = \rho_2 = 0$

Analog dengan cara pada kemungkinan II.1.a diperoleh penyelesaian dasar pertama yang berbentuk :

$$J_0(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(r+1)} \left(\frac{x}{2}\right)^{2r} \quad \ldots\ldots\ldots \text{ II-8}$$

Persamaan II-8 ini disebut fungsi Bessel macam pertama order = 0

Sedangkan penyelesaian dasar kedua diperoleh dengan menggunakan methode Frobenius.

Untuk n = 0 , Rumus Rekurensi II-2 berbentuk

$$C_{2r} = \frac{(-1)^r}{[(\rho+2)(\rho+4)\ldots\ldots(\rho+2s)]^2}\, C_0 \quad \ldots\ldots \text{ II-9}$$

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Selanjutnya dicari :

$$\frac{d\ln C_{2r}}{d\rho} = \frac{d}{d\rho} \ln(-1)^r C_0 - 2 \sum_{s=1}^{r} \ln(\rho + 2s)$$

$$= -2 \sum_{s=1}^{r} \frac{1}{\rho + 2s}$$

sehingga :

$$\frac{dC_{2r}}{d\rho} = C_{2r} \frac{d\ln C_{2r}}{d\rho}$$

$$= \frac{2(-1)^{r+1} C_0}{\prod_{s=1}^{r} (\rho + 2s)^2} \sum_{s=1}^{r} \frac{1}{\rho + 2s}$$

dan

$$\left.\frac{d\, C_{2r}}{d\rho}\right|_{\rho=0} = \frac{2(-1)^{r+1} C_0}{\prod_{s=1}^{r} (2s)^2} \sum_{s=1}^{r} \frac{1}{2s}$$

$$= \frac{(-1)^{r+1} C_0}{2^{2r} (r!)^2} \sum_{s=1}^{r} \frac{1}{s}$$

sedang :

$$\left.\frac{dC_{2r+1}}{d\rho}\right|_{\rho=0} = 0$$

Dengan mengambil :

$$C_0 = \frac{1}{2^0 \Gamma(1)} = 1$$

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maka didapat penyelesaian dasar kedua :

$$y_0 = J_0(x)\ \ln x + \sum_{r=0}^{\infty} \frac{(-1)^{r+1} \sum_{s=1}^{r} \frac{1}{s}}{(r!)^2} \left(\frac{x}{2}\right)^{2r}$$

.... II-10

Fungsi $Y_0(x)$ yang didefinisikan dengan persamaan II-10 dinamakan fungsi Bessel macam kedua order nol.

Dengan demikian penyelesaian umum dari Persamaan Defferensial II-1 adalah :

$$y = C_1\ J_0(x) + C_2\ Y_0(x)$$

Dimana C_1 dan C_2 adalah konstanta sebarang.

II.1.c. Ditinjau kemungkinan $\rho_1 \neq \rho_2$ dan

$$|\rho_1 - \rho_2| = 2\ n \text{ dengan } n \text{ bulat.}$$

Analog dengan cara pada kemungkinan II-1.a, diperoleh penyelesaian dasar pertama yang berbentuk :

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{2r+n} \quad \ldots\ \text{II-11}$$

Penyelesaian dasar kedua diperoleh dengan menggunakan metode Frobenius :

Dari rumus rekurensi II-2 didapat :

$$C_{2r} = \frac{(-1)^r}{\prod_{s=1}^{r} (2s+\rho+n)(2s+\rho-n)} C_0 \ldots\ldots\ldots\ldots \text{II-12}$$

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Bila diambil $C_0 = a(\rho - \rho_1)$ dan $\rho_1 = -n$

maka diperoleh :

$$C_{2r} = \frac{(-1)^r \; a(\rho + n)}{\prod\limits_{s=1}^{r} (2s + \rho + n)(2s + \rho - n)}$$

$$\ln . C_{2r} = \ln(-1)^r \, a + \ln(\rho + n) - \sum_{s=1}^{r} \left[\ln(2s + \rho + n) + \ln(2s - \rho - n) \right]$$

$$\frac{d \ln C_{2r}}{d\rho} = \frac{1}{\rho + n} - \sum_{s=1}^{r} \left\{ \frac{1}{2s + \rho + n} + \frac{1}{2s + \rho - n} \right\}$$

Dengan demikian :

$$\frac{d \; C_{2r}}{d\rho} = C_{2r} \frac{d \; \ln C_{2r}}{d\rho}$$

$$= \frac{(-1)^r . a}{\prod\limits_{s=1}^{r} (2s + \rho + n)(2s + \rho - n)} - \frac{(-1)^r \; a \; (\rho + n)}{\prod\limits_{s=1}^{r} (2s + \rho + n)(2s + \rho - n)} \cdot \sum_{s=1}^{r} \left\{ \frac{1}{2s + \rho + n} + \frac{1}{2s + \rho - n} \right\}$$

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untuk $r < n$

$$\left.\frac{d\ C_{2r}}{d\rho}\right|_{\rho=-n} = \frac{(-1)^r.a}{2^r.r!(-2)^r(n-1)\ldots(n-r)}$$

untuk $r = n$

$$\left.\frac{d\ C_{2r}}{d\rho}\right|_{\rho=-n} = \frac{a}{2^{2n}(n!)^2}$$

untuk $r > n$

$$\left.\frac{d\ C_{2r}}{d\rho}\right|_{\rho=-n} = \frac{(-1)^{r-n}\ .\ a}{2^{2r}\ .r!(n-1)\ \ (r-n)!}\ .$$

$$\left\{\sum_{s=0}^{r-n}\frac{1}{s+n}+\sum_{s=1}^{r-n}\frac{1}{s}\right\}$$

Dengan mengambil $C_0 = 2^{n-1}(n-1)!$ maka diperoleh penyelesaian dasar kedua.

$$y_n(x) = J_n(x)\ \ln x - \tfrac{1}{2}\sum_{r=0}^{n-1}\frac{(n-r-1)!}{r!}\left(\frac{x}{2}\right)^{-n+2r} -$$

$$\tfrac{1}{2}\left[\sum_{r=0}^{\infty}\frac{(-1)^r}{r!(n+r)!}\left(\frac{x}{2}\right)^{n+2r}\left\{\sum^{r+n}\frac{1}{s+n}\frac{1}{s}+\right.\right.$$

$$\left.\left.\sum_{s=1}^{r}\frac{1}{s}\right\}\right] \qquad \ldots\ldots\ldots\ldots\ \text{II-13}$$

Fungsi ini disebut fungsi Neuman Bessel macam kedua orde -n. Sehingga penyelesaian umum dari Persamaan Differensial II-1 adalah :

$$y = C_1\ J_n(x) + C_2\ \bar{y}_n(x) \qquad \ldots\ldots\ldots\ldots\ \text{II-14}$$

Dimana C_1 dan C_2 konstanta sebarang.

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II-2. RUMUS RECURENSI

Ditinjau penderetan dari :

$$\exp\left\{\tfrac{1}{2}x\left(t-\frac{1}{t}\right)\right\}=\exp(\tfrac{1}{2}\,x\,t)\,\exp\left(-\frac{x}{2t}\right)$$

Yaitu :

$$\exp\left\{\tfrac{1}{2}\,x\left(t-\frac{1}{t}\right)\right\}=\sum_{r=0}^{\infty}\frac{(xt)^r}{2^r\,.\,r!}\;\sum_{s=0}^{\infty}\frac{(-x)^s}{2^s.t^s.s!}$$

$$=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^s\left(\frac{x}{2}\right)^{r+s}\frac{t^{r-s}}{r!\;s!}$$

Substitusi index : $r - s = n$

maka didapat :

$$\exp\left\{\tfrac{1}{2}x\left(t-\frac{1}{t}\right)\right\}=\sum_{n=-\infty}^{\infty}\sum_{s=0}^{\infty}\frac{(-1)^s}{(n+s)!\,s!}$$

$$\left(\frac{x}{2}\right)^{n+2s}t^n \;\ldots\ldots\ldots\ldots \quad \text{II-15}$$

Dengan mengingat fungsi Bessel II- 5 dan II- 6 dapat ditulis :

$$\exp\left\{\tfrac{1}{2}x\left(t-\frac{1}{t}\right)\right\}=\sum_{n=-\infty}^{\infty}J_n(x)\;\;t^n \;\ldots\ldots \quad \text{II-16}$$

Terlihat bahwa $J_n(x)$ merupakan koeffisien dari t^n pada penderetan:

$$\exp\left\{\tfrac{1}{2}\,x\left(t-\frac{1}{t}\right).\right\}$$

Apabila kedua ruas dari II-16 dideferensialkan terhadap - x, maka didapat :

$$\tfrac{1}{2}\left(t-\frac{1}{t}\right)\exp\left\{\tfrac{1}{2}\,x\left(t-\frac{1}{t}\right)\right\}=\sum_{n=-\infty}^{\infty}J_n{}'(x)t^n$$

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$$\sum_{-\infty}^{\infty} \left[J_n(x)t^{n+1} - J_n(x)t^{n-1} \right] = \sum_{-\infty}^{\infty} 2\, J_n'(x)t^n$$

Bila pangkat dari pada t disamakan, diperoleh rumus :

$$J_{n-1}(x) - J_{n+1}(x) = 2\, J_n'(x) \;\ldots\ldots\ldots\ldots\; \text{II-17}$$

Apabila kedua ruas dari II-16 dideferensialkan terhadap t, maka diperoleh :

$$\tfrac{1}{2}\, x(1 + \frac{1}{t^2}) \exp \left\{ \tfrac{1}{2}x \left(t - \frac{1}{t}\right) \right\}$$

$$= \sum_{-\infty}^{\infty} n\, J_n(x)t^{n-1}$$

$$\sum_{-\infty}^{\infty} \left\{ \tfrac{1}{2}x\, J_n(x)t^n + \tfrac{1}{2}\, J_n(x)t^{n-2} \right\}$$

$$= \sum_{-\infty}^{\infty} n\, J_n(x)t^{n-1}$$

Bila pangkat dari t disamakan, diperoleh rumus :

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x}\, J_n(x) \;\ldots\ldots\ldots\ldots\; \text{II-18}$$

Bila persamaan II-17 ditambahkan pada persamaan - II-18, maka didapat :

$$x\, J_n'(x) = xJ_{n-1}(x) - nJ_n(x) \;\ldots\ldots\ldots\ldots\; \text{II-19}$$

Dan bila persamaan II-18 dikurangi dengan persamaan - II-17 didapat :

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$$x\, J_n'(x) = n\, J_n(x) - x\, J_{n+1}(x) \;\ldots\ldots\ldots\ldots\; \text{II-20}$$

Diambil keadaan khusus dari II-20, yaitu untuk n = 0 didapat hubungan :

$$J_0'(x) = -J_1(x) \qquad \ldots\ldots\ldots \quad \text{II-21}$$

Sedang untuk n = 1, persamaan II-19 menjadi

$$J_1'(x) = J_0(x) - \frac{1}{x} J_1(x) \qquad \ldots\ldots\ldots \quad \text{II-22}$$

Bila persamaan II-21 dideferensial terhadap x dan dengan-mengingat II-22 maka didapat hubungan :

$$J_0''(x) = -J_0(x) + \frac{1}{x} J_1(x) \qquad \ldots\ldots\ldots \quad \text{II-23}$$

Dengan mengingat definisi fungsi Bessel II-5 maka dapat - ditulis :

$$x^n J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \; x^{2n+2r}}{r!(n+r)!} (\tfrac{1}{2})^{n+2r} \ldots\ldots \quad \text{II-24}$$

Apabila kedua ruasnya dideferensialkan terhadap x, maka - diperoleh hubungan :

$$\frac{d}{dx}\left\{ x^n \; J_n(x) \right\} = \sum_{r=0}^{\infty} \frac{(-1)^r . x^{2n+2r-1}}{r!(n-1+r)!} (\tfrac{1}{2})^{n-1+2r}$$

$$= x^n \sum_{r=0}^{\infty} \frac{(-1)^r}{r!(n-1+r)!} \left(\frac{x}{2}\right)^{n-1+2r}$$

atau; dengan mengingat II-5 didapat :

$$\frac{d}{dx}\left\{ x^n J_n(x) \right\} = x^n \; J_{n-1}(x)$$

atau dapat pula ditulis sebagai :

$$\frac{1}{x} \; \frac{d}{dx}\left\{ x^n \; J_n(x) \right\} = x^{n-1} . J_{n-1}(x) \quad \ldots\ldots\ldots \; \text{II-25}$$

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Analog diatas, bila kedua ruas dari II-25 dideferensialkan terhadap x didapat :

$$\frac{d}{dx}\left[\frac{1}{x}\frac{d}{dx}\left\{x^n J_n(x)\right\}\right]= x^{n-1} J_{n-2}(x)$$

atau :

$$\left(\frac{1}{x}\frac{d}{dx}\right)^2\left\{x^n J_n(x)\right\}= x^{n-2} J_{n-2}(x) \ldots \qquad \text{II-26}$$

Untuk m bulat dan $m < n$ persamaan II-26 menjadi :

$$\left(\frac{1}{x}\frac{d}{dx}\right)^m\left\{x^n J_n(x)\right\}= x^{n-m} J_{n-m}(x) \ldots \qquad \text{II-27}$$

Dengan jalan yang sama, didapat

$$\left(\frac{1}{x}\frac{d}{dx}\right)\left\{x^{-n} J_n(x)\right\}= - x^{-(n+1)} J_{n+1}(x) \ldots \quad \text{II-28}$$

atau pada umumnya untuk m bulat persamaan II-28 menjadi :

$$\left(\frac{1}{x}\frac{d}{dx}\right)^m\left\{x^{-n} . J_n(x)\right\}= (-1)^m x^{-(n+m)} . J_{n+m}(x)$$

$$\ldots\ldots \quad \text{II-29}$$

Kejadian khusus dari II-29, yaitu bila diambil n = 0, maka diperoleh :

$$J_m(x) = (-1)^{-m} x^m \left(\frac{1}{x}\frac{d}{dx}\right)^m J_0(x) \ldots\ldots\ldots\ldots \quad \text{II-30}$$

Dari persamaan II-30 terlihat bahwa $J_m(x)$ dapat diturunkan dari $J_0(x)$.

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II-3. BENTUK INTEGRAL FUNGSI BESSEL

Ditinjau Integral :

$$I = \int_{-1}^{1} (1 - t^2)^{n-\frac{1}{2}} e^{ixt}\, dt \qquad \ldots\ldots\ldots\ldots\ldots \text{II-31}$$

dengan $n > -\frac{1}{2}$

Bila exponen (ixt) diperderetkan terhadap ixt, maka integral diatas menjadi :

$$I = \sum_{r=0}^{\infty} \frac{(ix)^r}{r!} \int_{-1}^{1} (1 - t^2)^{n-\frac{1}{2}} t^r \, dt \qquad \ldots\ldots\ldots \text{II-32}$$

untuk r ganjil integral diruas kanan sama dengan nol, sedang untuk r genap didapat :

$$2\int_{0}^{1} (1 - t^2)^{n-\frac{1}{2}} t^{2s}\, dt = \int_{0}^{1} (1 - u)^{n-\frac{1}{2}} u^{s-\frac{1}{2}} \, du$$

$$= \frac{\Gamma(n + \frac{1}{2})\, \Gamma(s + \frac{1}{2})}{\Gamma(n + s + 1)}$$

sehingga didapat :

$$I = \sum_{s=0}^{\infty} \frac{(-1)^s x^{2s}}{(2s)!} \frac{\Gamma(n + \frac{1}{2})\, \Gamma(s + \frac{1}{2})}{\Gamma(n+s+1)}$$

$$= \Gamma(\tfrac{1}{2})\, \Gamma(n+\tfrac{1}{2}) \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,(n+s)!} \left(\frac{x}{2}\right)^{2s} \qquad \ldots\ldots\ldots \text{II-33}$$

Dengan mengingat definisi fungsi Bessel akan diperoleh :

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$$J_n(x) = \frac{(\frac{1}{2}x)^n}{\Gamma(\frac{1}{2})\,\Gamma(n+\frac{1}{2})} \int_{-1}^{1} (1-t^2)^{n-\frac{1}{2}}\, e^{ixt}\, dt \qquad \text{II.34}$$

Karena exp (ixt) = Cos(xt) + i Sin(xt) dan mengingat fungsi Cos dan Sin berturut-turut merupakan fungsi-fungsi genap - dan ganjil, maka dengan substitusi t = Cos Θ, persamaan II-34 dapat ditulis sebagai :

$$J_n(x) = \frac{x^n}{2^{n-1}\,\Gamma(\frac{1}{2})\,\Gamma(n+\frac{1}{2})} \int_{0}^{\pi/2} \mathrm{Cos}(x\ \mathrm{Cos}\ \Theta)\mathrm{Sin}^{2n}\,\Theta\, d\Theta \ \ldots\ldots\ldots \text{II.35}$$

dan bila dipakai substitusi t = sinΘ, persamaan II.34 menjadi :

$$J_n(x) = \frac{x^n}{2^{n-1}\,\Gamma(\frac{1}{2})\,\Gamma(n+\frac{1}{2})} \int_{0}^{\pi/2} \mathrm{Cos}(x\mathrm{Sin}\Theta)\mathrm{Cos}^{2n}\,\Theta\, d\Theta \ \ldots \text{II.36}$$

Kejadian khusus yaitu bila n = 0, didapat :

$$J_0(x) = {}^2/_\pi \int_{0}^{\pi/2} \mathrm{Cos}(x\ \mathrm{Cos}\ \Theta)\, d\Theta$$

$$= {}^2/_\pi \int_{0}^{\pi/2} \mathrm{Cos}(x\ \mathrm{Sin}\ \Theta)\, d\Theta \ \ldots\ldots\ldots\ldots\ldots\ldots \text{II.37}$$

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II.4. RUMUS JUMLAHAN PADA KOEFISIEN BESSEL

Apabila pada persamaan II-16, Variabel x disubstitusikan dengan $x + y$ maka didapat :

$$e^{\left\{\frac{1}{2}(x+y)\left(t-\frac{1}{t}\right)\right\}} = \sum_{n=-\infty}^{\infty} J_n(x+y)\,t^n$$

sedang ruas kiri dapat ditulis sebagai :

$$e^{\left\{\frac{1}{2}x\left(t-\frac{1}{t}\right)\right\}}\; e^{\left\{\frac{1}{2}y\left(t-\frac{1}{t}\right)\right\}} =$$

$$\sum_{r=-\infty}^{\infty}\;\sum_{s=-\infty}^{\infty} J_r(x)\,J_s(y)\,t^{r+s}$$

Dengan demikian didapat hubungan :

$$J_n(x+y) = \sum_{s=-\infty}^{\infty} J_s(x)\,J_{n-s}(y) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad \text{II-38}$$

atau dapat ditulis :

$$J_n\,(x+y) = \sum_{r=-\infty}^{-1} J_r(x)\;J_{n-1}(y) \;+$$

$$\sum_{r=0}^{n} J_n(x)\;J_{n-r}(y) \;+$$

$$\sum_{r=n+1}^{\infty} J_r(x)\;J_{n-r}(y)$$

Dengan mengingat persamaan II-5 maka persamaan diatas menjadi :

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$$J_n(x+y) = \sum_{r=1}^{\infty} (-1)^r \; J_r(x) \; J_{n+r}(y) + \sum_{r=0}^{\infty} J_n(x) \; J_{n-r}(y) \; + \sum_{r=1}^{\infty} (-1)^r \; J_{n+r}(x) \; J_r(y)$$

atau

$$J_n(x+y) = \sum_{r=0}^{\infty} J_r(x) \; J_{n-r}(y) \; + \sum_{r=1}^{n} (-1)^r \left[J_r(x) J_{n+r}(y) + J_{n+r}(x) \; J_r(y) \right] \qquad \text{II-39}$$

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II.5. MODIFIKASI DARI FUNGSI BESSEL

Ditinjau Persamaan Diferensial :

$$\frac{d^2\Upsilon}{dy^2} + \frac{1}{y}\frac{d\Upsilon}{dy} - (m^2 + \frac{n^2}{y^2})\Upsilon = 0 \quad \ldots\ldots\ldots \text{II-40.}$$

Bila disubstitusikan variable x = my maka menjadi :

$$\frac{d^2\Upsilon}{dx^2} + \frac{1}{x}\frac{d\Upsilon}{dx} - (1 + \frac{n^2}{x^2})\Upsilon = 0 \quad \ldots\ldots \text{II-41}$$

Apabila persamaan ini diselesaikan dengan metode pen deretan, maka akan diperoleh penyelesaian berbentuk:

$$\Upsilon = C_1 I_n(x) + C_2 I_{-n}(x)$$

dengan C_1 dan C_2 konstanta sebarang dan

$$I_n(x) = \frac{x^n}{2^n\Gamma(n+1)} \sum_{r=0}^{\infty} \frac{(\frac{1}{4}x^2)^r}{r!(n+1)_r} \quad .\text{II-42}$$

$$= \frac{x^n}{2^n\Gamma(n+1)}\ {}_0F_1(n+1, \tfrac{1}{4}x^2)$$

dengan mengingat definisi fungsi Bessel, maka didapat hubungan :

$$I_n(x) = i^{-n} J_n(ix) \quad \ldots\ldots\ldots \text{II-43}$$

Apabila n bilangan bulat, maka $I_{-n}(x)$ dependent terhadap $I_n(x)$, sehingga penyelesaian kedua dari persamaan II-41 yang independent terhadap $I_n(x)$ dicari dengan metode Frabenius sehingga didapat :

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$$K_n(x) = I_n(x)\ln x - \tfrac{1}{2}\sum_{r=0}^{n-1}\frac{(n-r-1)!(-1)^r}{r!}\left(\frac{x}{2}\right)^{-n+2r}$$

$$-\tfrac{1}{2}\left[\sum_{r=0}^{\infty}\frac{(-1)^r}{r!(n+r)!}\left(\frac{x}{2}\right)^{n+2r}\left\{\sum_{s=n}^{r+n}\frac{1}{s} + \sum_{s-1}^{r}\frac{1}{s}\right\}\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{ II-44}$$

Fungsi-fungsi $I_n(x)$ dan $K_n(x)$ yang didefinisikan dengan persamaan II-42 dan II-44, berturut-turut disebut modifikasi - fungsi Bessel order -n macam pertama dan macam kedua.

Apabila n suatu bilangan bulat maka :

$$I_{-n}(x) = I_n(x) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{ II-45}$$

Modifikasi fungsi Bessel memenuhi pula rumus-rumus rekurensi untuk fungsi Bessel.

II-6. II-6.FUNGSI BER DAN BEI

Ditinjau persamaan Diferensial

$$\frac{d^2\Psi}{dy^2} + \frac{1}{y}\frac{d\Psi}{dy} - \frac{i\omega}{k} = 0 \quad \ldots\ldots\ldots\ldots \text{ II-46}$$

Bila diadakan substitusi variabel bebas

$x = \left(\frac{\omega}{k}\right)^{\frac{1}{2}} y$ maka persamaan diatas akan berubah menjadi :

$$\frac{d^2\Psi}{dy^2} + \frac{1}{x}\frac{d\Psi}{dx} - i\Psi = 0 \quad \ldots\ldots\ldots\ldots \text{ II-47}$$

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Merupakan kejadian khusus dari persamaan II-41 sehingga penyelesaiannya adalah :

$I_0\ (i^{\frac{1}{2}} x)$ dan $K_0(i^{\frac{1}{2}} x)$

Kemudian oleh Kelvin didefinisikan fungsi-fungsi baru sebagai bagian real dan imaginer dari :

$I_0(i^{\frac{1}{2}}x)$ dan $K_0(i^{\frac{1}{2}}x)$

yaitu :

$$I_0\ (i^{\frac{1}{2}}x) = ber(x) + i\ bei(x) \qquad \ldots\ldots\ \text{II-48}$$

$$K_0(i^{\frac{1}{2}}x) = Ker(x) + i\ Kei(x)$$

atau :

$$ber(x) = \sum_{s=0}^{\infty} \frac{(-1)^s (\frac{1}{4}x^2)^{2s}}{(2s!)^2}$$

$$bei(x) = \sum_{s=0}^{\infty} \frac{(-1)^s (\frac{1}{4}\ x^2)^{2s+1}}{((2s+1)!)^2} \qquad \ldots\ldots\ldots\ \text{II-49}$$

$$Ker(x) = ber(x)\ \ln x - \sum_{r=1}^{\infty} \frac{(-1)^r}{(2r)!^2} \left(\frac{x^2}{4}\right)^{2r+1} \sum_{s=1}^{2r} \left(\frac{1}{s}\right)$$

$$Kei(x) = bei(x)\ \ln x - \sum_{r=0}^{\infty} \frac{(-1)^r}{(2r+1)!^2} \left(\frac{x^2}{4}\right)^{2r+1} \sum_{s=1}^{2r+1} \left(\frac{1}{s}\right)$$

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II-7. INTEGRAL LOMMEL

Bila u dan v merupakan penyelesaian-penyelesaian dari persamaan diferensial Bessel :

$$x^2 \frac{d^2u}{dx^2} + x \frac{du}{dx} + (\lambda^2 x^2 - m^2)u = 0 \quad \ldots\ldots\ldots \text{II-50}$$

$$x^2 \frac{d^2v}{dx^2} + x \frac{dv}{dx} + (\mu^2 x^2 - n^2)v = 0 \quad \ldots\ldots\ldots \text{II-51}$$

Selanjutnya kedua persamaan diatas berturut-turut digandakan dengan $\frac{v}{x}$ dan $\frac{u}{x}$, kemudian diambil selisihnya; maka akan diperoleh hubungan :

$$xv\frac{d^2u}{dx^2} + v \frac{du}{dx} - xu\frac{d^2v}{dx^2} - u \frac{dv}{dx} +$$

$$\left\{(\lambda^2 - \mu^2)x + \frac{n^2 - m^2}{x}\right\} uv = 0$$

atau

$$\left\{(\lambda^2 - \mu^2)x + \frac{n^2 - m^2}{x}\right\}uv = xu \frac{d^2v}{dx^2} - xv \frac{d^2u}{dx^2}$$

$$+ u \frac{dv}{dx} - v \frac{du}{dx}$$

Kemudian kedua ruas diintegralkan terhadap x dari a sampai b; maka didapat :

$$\int_a^b \left\{ (\lambda^2 - \mu^2)x + \frac{n^2 - m^2}{x} \right\} uv \, dx =$$

$$\left[x \left(u \frac{dv}{dx} - v\frac{du}{dx}\right) \right]_a^b \quad \ldots\ldots\ldots \text{II-52}$$

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ambil $u = J_n(\lambda x)$ dan $v = J_m(\mu x)$

dan untuk $n = m$, didapat :

$$(\lambda^2 - \mu^2) \int_a^b x\, J_n(\lambda x)\, J_n(\mu x)\, dx =$$

$$x \left[\mu J_n(\lambda x)\, J_m'(\lambda x) - \lambda J_m(\mu x)\, J_n'(\mu x) \right]_a^b \quad \ldots \text{II-53}$$

Untuk $n > -1$

Persamaan II-52 disebut integral Lommel dari fungsi Bessel. Selanjutnya $\mu = \lambda + \varepsilon$ dengan ε kecil, maka penderetan Taylor terhadap ε, di sekitar $\varepsilon = 0$, dari fungsi $J_n(\mu x)$, adalah :

$$J_n(\mu x) = J_n\{(\lambda + \varepsilon)\, x\} =$$

$$J_n(\lambda x) + \varepsilon \frac{\partial}{\partial \varepsilon} J_n(\lambda x) + \ldots\ldots$$

Jika bentuk ini disubstitusikan pada II-52 akan diperoleh :

$$(-2\lambda\varepsilon - \varepsilon^2) \int_a^b x\, J_n(\lambda x) \left\{ J_n(\lambda x) + \varepsilon \frac{\partial}{\partial \varepsilon} J_n(\lambda x) + \ldots\ldots\ldots \right\} dx$$

$$= \varepsilon x \Big[J_n(\lambda x)\, J_n'(\lambda x) + \lambda x\, J_n(\lambda x)\, J_n''(x) - \lambda x \left\{ J_n'(\lambda x) \right\}^2 + \ldots\ldots\ldots \Big]_a^b$$

Bila kedua ruas dibagi dengan $-2\lambda\varepsilon$ dan diambil $\varepsilon \to 0$, dan dengan mengingat hubungan :

$$xJ_n''(x) + J_n'(x) = \frac{n^2}{x} J_n(x) - xJ_n(x)$$

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maka persamaan diatas akan menjadi :

$$\int_a^b x \left\{ J_n(\lambda x) \right\}^2 dx =$$

$$\frac{x^2}{2} \left[\left\{ J_n'(\lambda x) \right\}^2 - J_n(\lambda x)\, J_n''(\lambda x) - \right.$$

$$\left. \frac{1}{\lambda x} J_n(x)\, J_n'(x) \right]_a^b \qquad \ldots\ldots\ldots\ldots\ldots\ldots \text{II-54}$$

$$= \frac{x^2}{2} \left[\left\{ J_n'(\lambda x)^2 + (1 - \frac{n^2}{\lambda^2 x^2}) J_n(x) \right\}^2 \right]_a^b$$

Jika pada persamaan (II-49) diatas diambil $\lambda = \mu$ dan untuk $u = J_m(\lambda x)$ maka akan diperoleh hubungan :

$$\int_a^b J_m(\lambda x)\, J_n(\lambda x) \frac{dx}{x} - \frac{x}{n^2 - m^2} \left[J_m(\lambda x)\, J_n'(\lambda x) - \right.$$

$$\left. J_n(\lambda x)\, J_m'(\lambda x) \right]_a^b \ldots \text{II-55}$$

untuk $n > -1$

II-8. EXPANSI FOURIER BESSEL.

Andaikan f(x) suatu fungsi sembarang yang dapat diperderetkan sebagai :

$$f(x) = \sum_{m=1}^{\infty} C_m J_n(\lambda_m x) \qquad \ldots\ldots\ldots\ldots\ldots\ldots \text{II-56}$$

Untuk suatu interval $0 < x < a$, dan andaikan deret diruas - kanan konvergent didaerah (0,a) maka koefisien dari deret - ini dapat diperoleh dengan cara yang sama seperti pada penderetan FOURIER.

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Kedua ruas dari II-56 digandakan dengan $xJ_n(\lambda_p x)$, n =1,2,3,.. ..... kemudian diintegralkan terhadap x pada interval 0 - a, maka akan didapat :

$$\int_0^a x\, f(x)\, J_n(\lambda_p x)\, dx =$$

$$\sum_{m=1}^{\infty} \int_0^a C_m x\, J_n(\lambda_p x)\, J_n(\lambda_m x) dx \;\ldots\ldots\ldots\ldots \quad \text{II-57}$$

Dari persamaan diatas dapat dicari C_m, apabila fungsi-fungsi Bessel tersebut orthogonal terhadap "weight Function " pada interval 0 - a, yaitu bila :

$$\int_0^a x\, J_n(\lambda_p x)\, J_n(\lambda_m(x))\, dx = 0 \;\ldots\ldots\ldots\ldots \quad \text{II-58}$$

bila $\lambda_m \neq \lambda_p$

Integral Lommel II-53 memperlihatkan bahwa persamaan II-57 dipenuhi bila λ_m & λ_p memnuhi salah satu dari :

a. $J_n(\lambda_m a) = J_n(\lambda_p a) = 0$

b. $J_n(\lambda_m a) = J_n''(\lambda_p a) = 0$

c. $\lambda_p . J_n(\lambda_m a)\, J_n'(\lambda_p a) = J_m(\lambda_p a) J_n'(\lambda_m a)$

a,b,c II-59

Bila salah satu dari II-59 dipenuhi maka suku-suku dari II-57 adalah nol kecuali untuk $\lambda_m = \lambda_p$, dan koefisien C_m didapat - dari :

1. Bila dipenuhi syarat a, maka :

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$$C_m = \frac{2}{a^2 \left\{ J_n'(\ a\ ) \right\}^2} = \int_0^a x\ f(x)\ J_n(\lambda_m x)\ dx \quad \text{...II-60}$$

2. Bila dipenuhi syarat b , maka :

$$C_m = \frac{2}{a^2 . (1 - \frac{n^2}{\lambda_m^2 a^2})(J_n \lambda_m a)^2} \int_0^a x\ f(x)\ J_n(\lambda_m x)\ dx \quad \text{....II-61}$$

3. Bila dipenuhi syarat c, maka

$$C_m = \frac{2}{a^2 \left[\left\{ (J_n(\lambda_m a)^2 + (1 - \frac{n^2}{\lambda_m^2 a^2})\ (\ J_n(\lambda_m a)) \right\}^2 \right]} \int_0^a x\ f(x)\ J_n(\lambda_m x)\ dx \quad \text{.....................II-62}$$

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