

REDUCED-ORDER PARAMETER VARYING CONTROLLER WITH GUARANTEED CLOSED-LOOP PERFORMANCE

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Abstract. This paper presents an approach to order reduction of parameter varying controller. First, we find feasible solutions which satisfy parameter varying Lyapunov inequalities for constructing balanced parameter varying controller. Next, a singular perturbation method of time invariant systems is generalized to reduce the balanced controller order. Furthermore, we will show that the reduced-order of the parameter varying closed-loop system with the full-order controller is equivalent to the parameter varying closed-loop system with the reduced-order controller. We will also show that the reduced-order parameter varying controller guarantees closed-loop stability and performance. The effectiveness of the proposed controller reduction method is verified by applying it to a missile model. The closed-loop performance with the reduced-order parameter varying controller found by proposed method and that found by balanced truncation method is compared.

1. INTRODUCTION

Controller reduction of linear time invariant (LTI) systems has been published by Zhou, et al. [16]. They have reduced the order of LTI controller using balanced truncation (BT) method. Zobaidi, et al. [7] and Wood, et al. [15] generalized the BT method to reduce linear parameter varying systems (LPV) with unbounded parameter variation rates model and controller. They [7, 15] extend the twice of the sum of the tail formula well known in the LTI case. Further, Widowati, et al. [14] have used a generalization BT to reduce the order of a parameter varying

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controller. In paper [14], we have investigated a measure of the degradation of the parameter varying closed-loop performance due to the reduced-LPV controller.

Balanced singular perturbation approximation (BSPA) to reduce the controller order of LTI systems have been published by several authors [6, 10]. Widowati, et al. [11] generalized the BSPA method to reduce the model order of LPV system. Precisely, we have derived sufficient conditions for balanced and quadratic stability of the reduced-LPV system. In paper [12], we have derived the reduction error of BSPA method of quadratically stable LPV system. The upper bound of this reduction error was expressed in term of a \mathcal{L}_2 -norm bound. Widowati, et al. [13] have used a contractive right coprime factorization and BSPA approximation to reduce the model order of unstable LPV systems.

In this paper we propose a generalization of the BSPA method to reduce the LPV controller with unbounded parameter variation rates. The BSPA generalization is based on solutions of the parameter varying Lyapunov inequalities of the LPV system. We will show that the reduced-order LPV controller guarantees the LPV closed-loop stability and performance. In comparison with previous paper [14], this paper uses BSPA method, whereas in [14] BT method was used to reduce a high order-LPV controller. In BT method, the reduced-order LPV controller was obtained by truncating balanced states corresponding to smaller Q_e -singular values. In BSPA method, all balanced states are first divided into the slow and fast modes by defining the smaller Q_e -singular values as the fast mode, and the rest as the slow mode. Furthermore, a reduced-order parameter varying controller can be obtained by setting the velocity of the fast mode equal to zero. For all allowable parameter trajectories, the performance of the reduced-order LPV controller found by the BSPA method is compared with that found by BT method [14].

The paper is outlined as follows. Section 2 describes a problem formulation and some notations. Order reduction of LPV controller using BSPA method is presented in Section 3. Section 4 proposes the stability and performance of the closed-loop LPV system with the reduced-order LPV controller. In Section 5, the validity of the proposed method is demonstrated for a missile model. Finally, conclusions are given in Section 6.

2. PROBLEM STATEMENT

In this paper, we consider linear parameter varying systems. For a compact subset $\mathcal{P} \subset \mathcal{R}^s$, the parameter variation set \mathcal{F}_ρ denotes the set of all piecewise continuous mapping $\mathcal{R}(\text{time})$ into \mathcal{P} with a finite number of discontinuity in any interval. $\mathcal{F}_\rho := \{\rho(t) : \mathcal{R} \rightarrow \mathcal{P}, \rho_{i_{min}} \leq \rho_i \leq \rho_{i_{max}}, i = 1, 2, \dots, s\}$. A compact set $\mathcal{P} \subset \mathcal{R}^s$, along with continuous functions $A : \mathcal{R}^s \rightarrow \mathcal{R}^{n \times n}$, $B : \mathcal{R}^s \rightarrow \mathcal{R}^{n \times n_u}$, $C : \mathcal{R}^s \rightarrow \mathcal{R}^{n_v \times n}$, $D : \mathcal{R}^s \rightarrow \mathcal{R}^{n_v \times n_u}$ represent an n th-order parameter varying plant, $G(\rho)$, whose dynamics evolve as

$$\begin{aligned} \dot{x}(t) &= A(\rho(t))x(t) + B(\rho(t))u(t), \\ y(t) &= C(\rho(t))x(t) + D(\rho(t))u(t), \quad \forall \rho(t) \in \mathcal{F}_\rho. \end{aligned} \tag{1}$$

A state space realization of the parameter varying plant (1), $G(\rho)$, is written as

$$G(\rho) = \left[\begin{array}{c|c} A(\rho(t)) & B(\rho(t)) \\ \hline C(\rho(t)) & D(\rho(t)) \end{array} \right], \forall \rho(t) \in \mathcal{F}_\rho.$$

We assume that the full-order LPV controller has an m th-order. The parameter varying full-order controller has dynamics equation as follows.

$$\begin{aligned} \dot{x}_k(t) &= A_K(\rho)x_k(t) + B_K(\rho)y(t), \\ u(t) &= C_K(\rho)x_k(t), \quad \forall \rho(t) \in \mathcal{F}_\rho. \end{aligned} \quad (2)$$

The problem to be consider here is to find a reduced-order controller $\tilde{K}(\rho)$ by using BSPA with state space realization given by

$$\begin{aligned} \dot{x}_{k1}(t) &= \tilde{A}_k(\rho)x_{k1}(t) + \tilde{B}_k(\rho)y(t), \\ u(t) &= \tilde{C}_k(\rho)x_{k1}(t) + \tilde{D}_k(\rho)y(t), \quad \forall \rho(t) \in \mathcal{F}_\rho \end{aligned}$$

such that the degradation of the LPV closed-loop performance caused by applying a reduced-order controller is small, or at least bounded by a priori bound.

3. ORDER REDUCTION OF LPV CONTROLLER USING BSPA

In this section, we generalize some of the standard results on BSPA [6] of time invariant systems to controller reduction of the unbounded-rate parameter varying systems. This generalization is based on solutions of the parameter varying Lyapunov inequalities of the LPV system. Before reducing the order of LPV controller using BSPA, we give an example of the LPV system. For an LPV system (1), the parameter varying state space data can be demonstrated by the following example [5]. Suppose that the continuous matrix functions A and B are given as follows

$$A(t) := \begin{bmatrix} \omega^2(t)/M(t) & 1 \\ \cos(\theta(t)) & 1 + E(t) \end{bmatrix}, \quad B(t) = \begin{bmatrix} 0 \\ M(t) \end{bmatrix}.$$

Assume that the all continuous scalar-valued functions are bounded, and we can define

$$\begin{aligned} \rho_1(t) &:= \omega^2(t) && \in [\omega_{low}^2, \omega_{high}^2], \quad 0 < \omega_{low}^2 < \omega_{high}^2 < \infty \\ \rho_2(t) &:= M(t) && \in [M_{low}, M_{high}], \quad 0 < M_{low} < M_{high} < \infty, \\ \rho_3(t) &:= 1 + E(t) && \in [1 + E_{low}, 1 + E_{high}], \quad 0 < E_{low} < E_{high} < \infty, \\ \rho_4(t) &:= \cos(\theta(t)) && \in [-1, 1]. \end{aligned}$$

Then $\rho^T = [\rho_1, \rho_2, \rho_3, \rho_4]^T \in \mathcal{P} \subset \mathcal{R}^4$, where \mathcal{P} is defined by the bounds on ρ_1 - ρ_4 , and the LPV system matrices A and B become

$$A(\rho(t)) := \begin{bmatrix} \rho_1(t)/\rho_2(t) & 1 \\ \rho_4(t) & \rho_3(t) \end{bmatrix}, \quad B(\rho(t)) := \begin{bmatrix} 0 \\ \rho_2(t) \end{bmatrix}.$$

It can be seen from the above example that the parameter dependence of the linear system can be quite general.

Next, we introduce the concept of the quadratic stability of the LPV system. The parameter varying systems $G(\rho)$ is quadratically stable [5, 9, 15] if there exists a real positive-definite matrix $P = P^T > 0$ such that

$$A^T(\rho(t))P + PA(\rho(t)) < 0, \forall \rho(t) \in \mathcal{F}_\rho. \quad (3)$$

The induced \mathcal{L}_2 norm of a quadratically stable parameter varying systems, $G(\rho)$, with zero initial conditions, is defined as [3]

$$\|G(\rho)\|_{i,2} := \sup_{\rho(t) \in \mathcal{F}_\rho} \sup_{u \neq 0, u \in \mathcal{L}_2} \frac{\|y\|_2}{\|u\|_2}.$$

To formulate a performance oriented parameter varying output feedback synthesis problem, the parameter varying plant is rewritten as follows

$$\begin{aligned} \dot{x}(t) &= A(\rho(t))x(t) + B_1(\rho(t))w(t) + B_2(\rho(t))u(t), \\ z(t) &= C_1(\rho(t))x(t) + D_{12}(\rho(t))u(t), \\ y(t) &= C_2(\rho(t))x(t) + D_{21}(\rho(t))w(t), \quad \forall \rho(t) \in \mathcal{F}_\rho. \end{aligned} \quad (4)$$

A construction of full-order LPV controller (2) has been developed by Apkarian, et al. [1]. The design objectives are to satisfy \mathcal{H}_∞ performance criterion, i.e., the parameter varying closed-loop systems is quadratically stable over \mathcal{P} and the \mathcal{L}_2 gain of the parameter varying closed-loop systems is bounded by γ , $\gamma > 0$ for all possible trajectories ρ .

The parameter varying closed-loop system $T_{zw}(\rho)$ with the full-order LPV controller can be expressed as follows. (For simplicity, the dependence of parameter ρ on t is omitted)

$$T_{zw}(\rho) = \left[\begin{array}{c|c} \bar{A}(\rho) & \bar{B}(\rho) \\ \hline \bar{C}(\rho) & 0 \end{array} \right], \quad \forall \rho \in \mathcal{F}_\rho, \quad (5)$$

where

$$\bar{A}(\rho) = \begin{bmatrix} A(\rho) & B_2 C_K(\rho) \\ B_K(\rho) C_2(\rho) & A_K(\rho) \end{bmatrix}, \quad \bar{B}(\rho) = \begin{bmatrix} B_1(\rho) \\ B_K(\rho) D_{21}(\rho) \end{bmatrix},$$

$$\bar{C}(\rho) = [C_1(\rho) \quad D_{12}(\rho) C_K(\rho)].$$

Assume that the full-order LPV controller is a stabilizing parameter varying controller so that $T_{zw}(\rho)$ is quadratically stable over \mathcal{P} and that there exist $U = \text{diag}(U_1, U_2) > 0$ and $V = \text{diag}(V_1, V_2) > 0$ such that $\forall \rho \in \mathcal{F}_\rho$,

$$\bar{A}(\rho)U + U\bar{A}^T(\rho) + \bar{B}(\rho)\bar{B}^T(\rho) < 0, \quad (6)$$

$$\bar{A}^T(\rho)V + V\bar{A}(\rho) + \bar{C}^T(\rho)\bar{C}(\rho) < 0. \quad (7)$$

Necessary and sufficient conditions for the solutions existence of above inequalities are a quadratic stability of plant $G(\rho)$ [9]. Assume that there exist nonsingular matrices T_1 and T_2 such that,

$$T_1 U_1 T_1^T = T_1^{-T} V_1 T_1^{-1} = \Sigma_g = \text{diag}(\xi_1, \xi_2, \dots, \xi_n), \quad \xi_1 \geq \xi_2 \geq \dots \geq \xi_n$$

and

$$T_2 U_2 T_2^T = T_2^{-T} V_2 T_2^{-1} = \Sigma_k = \text{diag}(\Sigma_{k1}, \Sigma_{k2}),$$

with $\Sigma_{k1} = \text{diag}(\sigma_1, \dots, \sigma_r)$, $\Sigma_{k2} = \text{diag}(\sigma_{r+1}, \dots, \sigma_m)$, $\sigma_r > \sigma_{r+1}$, and $\sigma_j \geq \sigma_{j+1}$, $j = 1, 2, \dots, m$ are \mathcal{Q} -singular values of $K(\rho)$. The balanced state space realization $K(\rho)$ can be written as

$$\begin{aligned} K(\rho) &= \left[\begin{array}{c|c} T_2 A_K(\rho) T_2^{-1} & T_2 B_K(\rho) \\ \hline C_K(\rho) T_2^{-1} & 0 \end{array} \right], \\ &= \left[\begin{array}{c|c} A_k(\rho) & B_k(\rho) \\ \hline C_k(\rho) & 0 \end{array} \right], \quad \forall \rho \in \mathcal{F}_\rho. \end{aligned}$$

Further, $K(\rho)$ is partitioned conformably with $\Sigma_k = \text{diag}(\Sigma_{k1}, \Sigma_{k2})$ as

$$K(\rho) = \left[\begin{array}{cc|c} A_{k11}(\rho) & A_{k12}(\rho) & B_{k1}(\rho) \\ A_{k21}(\rho) & A_{k22}(\rho) & B_{k2}(\rho) \\ \hline C_{k1}(\rho) & C_{k2}(\rho) & 0 \end{array} \right]. \quad (8)$$

where $A_{k1}(\rho)$ and $\Sigma_1 \in \mathcal{R}^{r \times r}$. The dynamics equation of balanced full-order LPV controller (8) can be written as

$$\begin{aligned} \begin{bmatrix} \dot{x}_{k1}(t) \\ \dot{x}_{k2}(t) \end{bmatrix} &= \begin{bmatrix} A_{k11}(\rho) & A_{k12}(\rho) \\ A_{k21}(\rho) & A_{k22}(\rho) \end{bmatrix} \begin{bmatrix} x_{k1}(t) \\ x_{k2}(t) \end{bmatrix} + \begin{bmatrix} B_{k1}(\rho) \\ B_{k2}(\rho) \end{bmatrix} y(t), \\ u(t) &= [C_{k1}(\rho) \quad C_{k2}(\rho)] \begin{bmatrix} x_{k1}(t) \\ x_{k2}(t) \end{bmatrix}, \end{aligned} \quad (9)$$

where $x_{k1}(t) \in R^r$, $x_{k2}(t) \in R^{m-r}$. When the system is balanced, states corresponding to \mathcal{Q}_e -smaller singular values represent the fast dynamics of the systems (i.e. its states have very fast transient dynamics and decay rapidly to certain *steady value*). Based on the concept of the BSPA method [6, 10], the system (9) can be approximated by setting $\dot{x}_{k2}(t) = 0$, so that (9) becomes

$$\begin{aligned} \dot{x}_{k1}(t) &= A_{k11}(\rho)x_{k1}(t) + A_{k12}(\rho)x_{k2}(t) + B_{k1}y(t), \\ 0 &= A_{k21}(\rho)x_{k1}(t) + A_{k22}(\rho)x_{k2}(t) + B_{k2}(\rho)y(t), \\ u(t) &= C_{k1}(\rho)x_{k1}(t) + C_{k2}(\rho)x_{k2}(t). \end{aligned}$$

The dynamics of the reduced-order LPV controller can be expressed as

$$\begin{aligned} \dot{x}_{k1}(t) &= \tilde{A}_{k1}(\rho)x_{k1}(t) + \tilde{B}_k(\rho)y(t), \\ u(t) &= \tilde{C}_k(\rho)x_{k1}(t) + \tilde{D}_k(\rho)y(t), \end{aligned} \quad (10)$$

where

$$\tilde{A}_k(\rho) = A_{k11}(\rho) - A_{k12}(\rho)A_{k22}^{-1}(\rho)A_{k21}(\rho), \quad (11)$$

$$\tilde{B}_k(\rho) = B_{k1}(\rho) - A_{k12}(\rho)A_{k22}^{-1}(\rho)B_{k2}(\rho), \quad (12)$$

$$\tilde{C}_k(\rho) = C_{k1}(\rho) - C_{k2}(\rho)A_{k22}^{-1}(\rho)A_{k21}(\rho), \quad (13)$$

$$\tilde{D}_k(\rho) = -C_{k2}(\rho)A_{k22}^{-1}(\rho)B_{k2}(\rho). \quad (14)$$

4. PERFORMANCE OF THE REDUCED-ORDER LPV CONTROLLER

The aim of the LPV controller reduction is not only to obtain a stabilizing reduced-order controller but also to ensure that the performance of closed-loop system with the reduced-order LPV controller is as close as possible to that of the closed-loop system with the full-order LPV controller. In this section, we show that the degradation of the performance of LPV system caused by applying the reduced-order controller is bounded by a priori bound.

Define $T_t = \text{diag}(T_1, T_2)$ and $\Sigma_t = \text{diag}(\Sigma_{t1}, \Sigma_{t2})$ with $\Sigma_{t1} = \text{diag}(\xi_1, \xi_2, \dots, \xi_n, \sigma_1, \sigma_2, \dots, \sigma_r)$ and $\Sigma_{t2} = \Sigma_{k2} = \text{diag}(\sigma_{r+1}, \dots, \sigma_m)$. Then the parameter varying closed-loop system

$$\bar{T}_{zw}(\rho) = \left[\begin{array}{c|c} T_t \bar{A}(\rho) T_t^{-1} & T_t \bar{B}(\rho) \\ \hline \bar{C}(\rho) T_t^{-1} & 0 \end{array} \right] = \left[\begin{array}{c|c} A_t(\rho) & B_t(\rho) \\ \hline C_t(\rho) & 0 \end{array} \right] \quad (15)$$

is balanced, such that $\forall \rho \in \mathcal{F}_\rho$,

$$A_t(\rho)\Sigma_t + \Sigma_t A_t^T(\rho) + B_t(\rho)B_t^T(\rho) < 0,$$

$$A_t^T(\rho)\Sigma_t + \Sigma_t A_t(\rho) + C_t^T(\rho)C_t(\rho) < 0.$$

The parameter varying closed-loop system (15), $\bar{T}_{zw}(\rho)$, is partitioned conformably with $\text{diag}(\Sigma_{t1}, \Sigma_{t2})$ as

$$\begin{aligned} A_t(\rho) &= \begin{bmatrix} A_{t11}(\rho) & A_{t12}(\rho) \\ A_{t21}(\rho) & A_{t22}(\rho) \end{bmatrix} \\ &= \begin{bmatrix} T_1 A(\rho) T_1^{-1} & T_1 B_2(\rho) C_{k1}(\rho) & T_1 B_2(\rho) C_{k2}(\rho) \\ B_{k1}(\rho) C_2(\rho) T_1^{-1} & A_{k11}(\rho) & A_{k12}(\rho) \\ B_{k2}(\rho) C_2(\rho) T_1^{-1} & A_{k21}(\rho) & A_{k22}(\rho) \end{bmatrix}, \end{aligned} \quad (16)$$

$$B_t = \begin{bmatrix} B_{t1}(\rho) \\ B_{t2}(\rho) \end{bmatrix} = \begin{bmatrix} T_1 B_1(\rho) \\ B_{k1}(\rho) D_{21}(\rho) \\ B_{k2}(\rho) D_{21}(\rho) \end{bmatrix}, \quad (17)$$

$$C_t = [C_{t1}(\rho) \quad C_{t2}(\rho)] = [C_1(\rho) T_1^{-1} \quad D_{12}(\rho) C_{k1}(\rho) \quad D_{12}(\rho) C_{k2}(\rho)]. \quad (18)$$

The reduced-order LPV system by singular perturbation of the balanced LPV closed-loop realization $(\mathcal{F}_\rho, A_t(\rho), B_t(\rho), C_t(\rho), D_t(\rho))$ is

$$\begin{aligned} \tilde{T}_{zw} &= \left[\begin{array}{c|c} \tilde{A}_t(\rho) & \tilde{B}_t(\rho) \\ \hline \tilde{C}_t(\rho) & \tilde{D}_t(\rho) \end{array} \right] \\ &= \left[\begin{array}{c|c} A_{t11}(\rho) - A_{t12}(\rho)A_{t22}^{-1}(\rho)A_{t21}(\rho) & B_{t1}(\rho) - A_{t12}(\rho)A_{t22}^{-1}(\rho)B_{t2}(\rho) \\ \hline C_{t1}(\rho) - C_{t2}(\rho)A_{t22}^{-1}(\rho)A_{t21}(\rho) & -C_{t2}(\rho)A_{t22}^{-1}(\rho)B_{t2}(\rho) \end{array} \right]. \end{aligned} \quad (19)$$

From the equations (16)-(18), and (19), we get

$$\begin{aligned} \tilde{A}_t(\rho) &= \left[\begin{array}{c} T_1\{A(\rho) - B_2(\rho)C_{k2}(\rho)A_{k22}^{-1}(\rho)B_{k2}C_2(\rho)\}T_1^{-1} \\ \{B_{k1}(\rho) - A_{k12}(\rho)A_{k22}^{-1}(\rho)B_{k2}(\rho)\}C_2(\rho)T_1^{-1} \\ \\ T_1B_2(\rho)\{C_{k1}(\rho) - C_{k2}(\rho)A_{k22}^{-1}(\rho)A_{k21}(\rho)\} \\ A_{k11}(\rho) - A_{k12}(\rho)A_{k22}^{-1}(\rho)A_{k21}(\rho) \end{array} \right] \\ \tilde{B}_t(\rho) &= \left[\begin{array}{c} T_1\{B_1(\rho) - B_2(\rho)C_{k2}(\rho)A_{k22}^{-1}(\rho)B_{k2}(\rho)D_{21}(\rho)\} \\ \{B_{k1}(\rho) - A_{k12}(\rho)A_{k22}^{-1}(\rho)B_{k2}(\rho)\}D_{21}(\rho) \end{array} \right], \\ \tilde{C}_t(\rho) &= [\{C_1(\rho) - D_{12}(\rho)C_{k2}(\rho)A_{k22}^{-1}(\rho)B_{k2}(\rho)C_2(\rho)\}T_1^{-1} \\ &\quad D_{12}(\rho)\{C_{k1}(\rho) - C_{k2}(\rho)A_{k22}^{-1}(\rho)A_{k21}(\rho)\}], \\ \tilde{D}_t(\rho) &= -D_{12}(\rho)C_{k2}(\rho)A_{k22}^{-1}(\rho)B_{k2}(\rho)D_{21}(\rho). \end{aligned}$$

The properties of the singular perturbation reduction method of the balanced LPV system are given in the next results (Widowati, et al., 2003) [11, 12].

Theorem 4.1. [11] *Let $(\mathcal{F}_\rho, A(\rho), B(\rho), C(\rho), D(\rho))$ be balanced realization with the Gramian Σ of the LPV system and quadratically stable. Then the singular perturbation approximation realization $(\mathcal{F}_\rho, \tilde{A}(\rho), \tilde{B}(\rho), \tilde{C}(\rho), \tilde{D}(\rho))$ is balanced with Gramian Σ_1 and quadratically stable.*

Theorem 4.2. [12] *Assume $G_n(\rho) = \begin{bmatrix} A_{11}(\rho) & A_{12}(\rho) & B_1(\rho) \\ A_{21}(\rho) & A_{22}(\rho) & B_2(\rho) \\ C_1(\rho) & C_2(\rho) & D(\rho) \end{bmatrix}$, the n^{th} -order LPV system, is quadratically stable, \mathcal{Q}_e -balanced, and partitioned conformably with $\Sigma = \text{diag}(\Sigma_1, \Sigma_2)$. Then the error bound for the singular perturbation approximation $\tilde{G}_r(\rho)$ of LPV system $G(\rho)$ is $\|G_n(\rho) - \tilde{G}_r(\rho)\|_{i,2} \leq 2 \sum_{j=r+1}^n \sigma_j$.*

In the next theorem, we show that the reduced-order of the LPV closed-loop system with the full-order LPV controller is equivalent to the LPV closed-loop system with the reduced-order LPV controller using BSPA (10). This theorem

also shows that the singular perturbation approximation of $K(\rho)$ guarantees LPV closed-loop stability and performance.

Theorem 4.3. *Assume that a reduced-order LPV controller using BSPA is given by*

$$\tilde{K}(\rho) = \left[\begin{array}{c|c} \tilde{A}_k(\rho) & \tilde{B}_k(\rho) \\ \hline \tilde{C}_k(\rho) & \tilde{D}_k(\rho) \end{array} \right]$$

and $\tilde{T}_{zwq}(\rho)$ is a LPV closed loop system with reduced-order controller $\tilde{K}(\rho)$. Then $\tilde{T}_{zwq}(\rho) = \tilde{T}_{zw}(\rho)$, $\tilde{T}_{zwq}(\rho)$ is quadratically stable, and $\|T_{zw} - \tilde{T}_{zwq}(\rho)\|_{i,2} \leq 2 \sum_{j=r+1}^m \sigma_j$.

Proof. To investigate that $\tilde{T}_{zwq}(\rho) = \tilde{T}_{zw}(\rho)$, it is sufficient to show that a realization of $\tilde{T}_{zwq}(\rho)$ the same as that of $\tilde{T}_{zw}(\rho)$. The LPV closed-loop system using the reduced-order LPV controller $\tilde{K}(\rho)$ and balanced plant with T_1 has realization $(\mathcal{F}_\rho, \tilde{A}_q(\rho), \tilde{B}_q(\rho), \tilde{C}_q(\rho), \tilde{D}_q(\rho))$, where

$$\tilde{A}_q(\rho) = \left[\begin{array}{cc} T_1 \{A(\rho) + B_2(\rho)\tilde{D}_k(\rho)C_2(\rho)\}T_1^{-1} & T_1 B_2(\rho)\tilde{C}_k(\rho) \\ \tilde{B}_k(\rho)C_2(\rho)T_1^{-1} & \tilde{A}_k(\rho) \end{array} \right], \quad (20)$$

$$\tilde{B}_q(\rho) = \left[\begin{array}{c} T_1 \{B_1(\rho) + B_2(\rho)\tilde{D}_k(\rho)D_{21}(\rho)\} \\ \tilde{B}_k(\rho)D_{21}(\rho) \end{array} \right], \quad (21)$$

$$\tilde{C}_q(\rho) = [(C_1(\rho) + D_{12}(\rho)\tilde{D}_k(\rho)C_2(\rho))T_1^{-1} \quad D_{12}(\rho)\tilde{C}_k(\rho)], \quad (22)$$

$$\tilde{D}_q(\rho) = D_{12}(\rho)\tilde{D}_k(\rho)D_{21}(\rho), \quad (23)$$

and realization $(\mathcal{F}_\rho, \tilde{A}_k(\rho), \tilde{B}_k(\rho), \tilde{C}_k(\rho), \tilde{D}_k(\rho))$ is equal to (11)-(14). Furthermore, we will show that the realizations of

$$\tilde{T}_{zwq}(\rho) = \left[\begin{array}{c|c} \tilde{A}_q(\rho) & \tilde{B}_q(\rho) \\ \hline \tilde{C}_q(\rho) & \tilde{D}_q(\rho) \end{array} \right]$$

and

$$\tilde{T}_{zw}(\rho) = \left[\begin{array}{c|c} \tilde{A}_t(\rho) & \tilde{B}_t(\rho) \\ \hline \tilde{C}_t(\rho) & \tilde{D}_t(\rho) \end{array} \right]$$

are equal to each other, as follows.

(i) For each state space matrix of $\tilde{A}_q(\rho)$, it can be written as (1×1)

$$\tilde{D}_k(\rho) = -C_{k2}(\rho)A_{k22}^{-1}(\rho)B_{k2}(\rho), \quad (24)$$

(1×2)

$$\tilde{C}_k(\rho) = C_{k1}(\rho) - C_{k2}(\rho)A_{k22}^{-1}(\rho)A_{k21}(\rho), \quad (25)$$

(2×1)

$$\tilde{B}_k(\rho) = B_{k1}(\rho) - A_{k12}(\rho)A_{k22}^{-1}(\rho)B_{k2}(\rho), \quad (26)$$

(2×2)

$$\tilde{A}_k(\rho) = A_{k11}(\rho) - A_{k12}(\rho)A_{k22}^{-1}(\rho)A_{k21}(\rho). \quad (27)$$

Now, substituting the above equations (24)-(27) into (20)-(23), we get $\tilde{A}_q(\rho) = \tilde{A}_t(\rho)$.

(ii) Substitution of (24) and (26) in (21) gives $\tilde{B}_q(\rho) = \tilde{B}_t(\rho)$.

(iii) Using (24), (25), and (22), it is found that $\tilde{C}_q(\rho) = \tilde{C}_t(\rho)$.

(iv) Substitution of (24) in (23) yields $\tilde{D}_q(\rho) = \tilde{D}_t(\rho)$.

From (i)-(iv) and based on Theorem 4.1 and Theorem 4.2, we obtain that $\tilde{T}_{zwq}(\rho) = \tilde{T}_{zw}(\rho)$, \tilde{T}_{zwq} is quadratically stable $\forall \rho \in \mathcal{F}_\rho$, and $\|T_{zw}(\rho) - \tilde{T}_{zwq}(\rho)\|_\infty \leq 2tr(\Sigma_{t2}) = 2 \sum_{j=r+1}^m \sigma_j$. \blacksquare

Solutions U and V which satisfy inequalities (6) and (7) are not unique. Non-uniqueness of U, V can be exploited to produce more desirable reduced-order. Considering the degradation of the closed-loop performance $\|T_{zw}(\rho) - \tilde{T}_{zwq}(\rho)\|_{i,2} \leq 2 \sum_{i=r+1}^m \sigma_i$ given in the Theorem 4.3, it is reasonable to choose U and V such that $\sum_{i=r+1}^m \sigma_i = \sum_{i=r+1}^m \lambda_i^{1/2}(U_2 V_2)$ is minimized. However, such solutions are hard to compute since the corresponding objective function to be optimized is not convex. In this paper we mainly consider the convex objective function, hence the solutions is not guaranteed to be optimal, commonly called as suboptimal solutions.

Note from (5)-(7) that $A(\rho)$ and $A_K(\rho)$ are quadratically stable over \mathcal{P} . These indicate that the parameter varying plant and controller are quadratically stable. Hence the previous results are not applicable to unstable parameter varying plant and/or unstable parameter varying controller. If $G(\rho)$ and/or $K(\rho)$ are not quadratically stable, then we use coprime factorization of the parameter varying controller. This is constructed by using feasible solutions of linear matrix inequality with a procedure as follows. Define right coprime factorization (RCF) of the $K(\rho) = \mathcal{U}(\rho)\mathcal{V}(\rho)^{-1}$,

$$\begin{bmatrix} \mathcal{U}(\rho) \\ \mathcal{V}(\rho) \end{bmatrix} = \left[\begin{array}{c|c} A_K(\rho) + B_K(\rho)F_K(\rho) & B_K(\rho) \\ \hline C_K(\rho) & 0 \\ F_K(\rho) & I \end{array} \right],$$

where $\mathcal{U}(\rho)$ and $\mathcal{V}(\rho)$ are quadratically stable, $F_K(\rho) = -B_K(\rho)^T X$, $X = X^T > 0$ is a feasible solution of the inequality: $X A_K(\rho) + A_K(\rho)^T X - X B_K(\rho) B_K(\rho)^T X < 0$, $\forall \rho \in \mathcal{F}_\rho$. Let Q and P are observability and controllability Gramians of RCF, where $P = P^T > 0$ and $Q = Q^T > 0$ solve Lyapunov inequalities:

$(A_K(\rho) + B_K(\rho)F_K(\rho))P + P(A_K(\rho) + B_K(\rho)F_K(\rho))^T + B_K(\rho)B_K(\rho)^T < 0$,
 $(A_K(\rho) + B_K(\rho)F_K(\rho))^T Q + Q(A_K(\rho) + B_K(\rho)F_K(\rho)) + C_K(\rho)^T C_K(\rho) + F_K(\rho)^T F_K(\rho) < 0$, $\forall \rho \in \mathcal{F}_\rho$. Furthermore, construct balanced $K(\rho)$ by using a state transformation matrix such that the transformed Gramians $P = Q = \Sigma = \text{diag}(\sigma_1, \dots, \sigma_m)$ and then apply BSPA method to obtain $\hat{K}(\rho)_r = \mathcal{U}(\rho)_r \mathcal{V}(\rho)_r^{-1}$. $\hat{K}(\rho)_r$ has r th-order, with $r < m$.

5. CASE STUDY

In this section, a comparison of the performance on the missile vertical acceleration of the reduced parameter varying controllers obtained by balanced truncation [14] and BSPA method is presented. The missile dynamics [2, 3] under consideration vary greatly as a function of speed (v), angle of attack (α), and altitude (H). The parameter varying system $G(\rho)$ is described as follows.

$$\begin{bmatrix} \dot{\alpha} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} -Z_\alpha & 1 \\ -M_\alpha & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ q \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \delta_m,$$

$$\begin{bmatrix} a_{zv} \\ q \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ q \end{bmatrix},$$

where $\rho_1 = Z_\alpha$ and $\rho_2 = M_\alpha$ are parameter depending on v , α , and H . We assume that ρ_1 and ρ_2 are measured in real time. The states are angle of attack (α) and pitch rate (q). The input to the model is fin deflection (δ_m). The output are normalized vertical acceleration (a_{zv}) and pitch rate (q). Let v , α , and H vary in $0.5 \leq v \leq 4$ (mach), $0 \leq \alpha \leq 40$ (degrees), $0 \leq H \leq 18$ (km) during operation, the coefficients ρ_1 and ρ_2 range in $\rho_1 \in [0.5, 4]$ dan $\rho_2 \in [0, 106]$. Note that ρ_1 and ρ_2 enter the states space matrices in an affine way and it can be considered as representing a convex polytope with four vertices. The aim of the controller design is to achieve the step response of the vertical acceleration (a_{zv}) having settling time < 0.5 seconds for all variations of ρ_1 and ρ_2 and overshoot limitation is 15%. The selection of weights is based on a frozen-time analysis of the parameter varying systems, and follows the same lines as \mathcal{H}_∞ . The weights must enforce adequate settling time 0.5 seconds and high-frequency gain attenuation [2, 3, 4]. We choose the sensitivity weight W_1 and the robustness weight W_2 . The performance and robustness requirements correspond to designing a controller which satisfy

$$\left\| \begin{bmatrix} W_1(I + G_\rho K_\rho)^{-1} \\ W_2 K_\rho(I + G_\rho K_\rho)^{-1} \end{bmatrix} \right\|_\infty < 1,$$

where

$$W_1 = \frac{2.01}{s+0.201}, \quad W_2 = \frac{96.78s^3 + 0.29s^2 + 0.0003s - 0.0039}{s^3 + 1.12 \times 10^6 s^2 + 1.05 \times 10^8 s + 1.07 \times 10^{11}}.$$

The order of the generalized plant (4) is 6. The full-order parameter varying controller is designed by using procedure suggested in Section 3. The resulting parameter varying controller has the same order as that of the generalized plant. The feasibility problems (6) and (7) are convex problems. Using projection algorithm [8] implemented in LMI Control toolbox for MATLAB running on pentium(R) 4, 2400 MHz, 18x, 512 MB of RAM, we obtain feasible solutions U and V after 78 iterations for convergence. This corresponds to CPU time of 2.157 seconds. The evolution of convergence values during the alternate iterations for calculating U and V solutions is given in Figure 1. Furthermore, the order of the parameter varying controller is reduced by using the BT method [14] and proposed (BSPA) method. The upper bound of the parameter varying closed-loop performance for

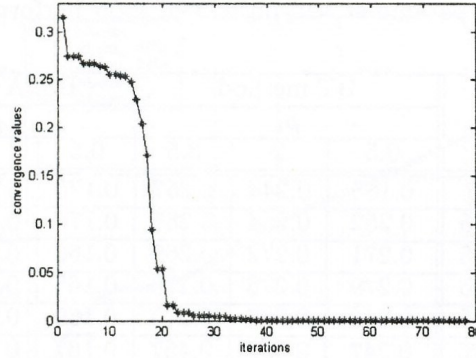


Figure 1: Evolution of convergence values versus iterations

all variations of parameters is 0.267. At 15 frozen parameters, \mathcal{H}_∞ -norm of the parameter varying closed loop systems with full-order and some reduced-order controllers are shown in Table 1 and Table 2. From Table 2, it can be seen generally that the \mathcal{H}_∞ -norm of the closed loop systems with reduced-controller obtained by proposed method is lower than that found by BT method. If the full-order parameter varying controller is reduced by using BT method up to 4th-order and by proposed method up to 3th-order, then \mathcal{H}_∞ -norm of the closed loop systems turn

Table 1. The frozen parameter varying closed loop performance with full-order controller

Full-order		ρ_1		
		0.5	2	3.5
ρ_2	5	0.175	0.167	0.174
	25	0.170	0.167	0.178
	40	0.166	0.171	0.182
	65	0.165	0.175	0.185
	85	0.169	0.179	0.189

out be much greater than \mathcal{H}_∞ -norm of the closed loop with full-order controller.

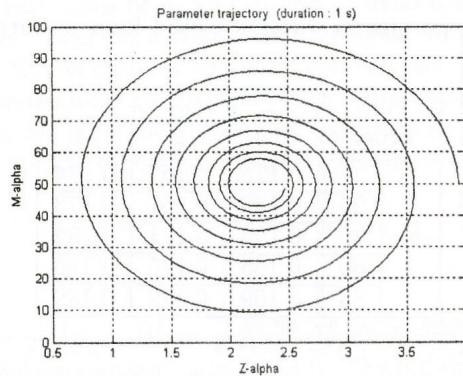
To analyze stability robustness and performance of the closed-loop systems we use the particular spiral parameter trajectory [15] as follows.

$$\begin{aligned}\rho_1(t) &= 2.25 + 1.70e^{-4t}\cos(100t), \\ \rho_2(t) &= 50 + 49e^{-4t}\sin(100t).\end{aligned}$$

This trajectory is depicted in Figure 2. To conform the quadratic stability of the full- and reduced-closed loop system, the solution P satisfying inequality (3) is found after 47, 37, 39 iterations for 6th-, 5th-, 4th-order controllers using BSPA. The step responses of the LPV closed-loop system with the 6th, 5th, and 4th-order parameter

Table 2. The frozen parameter varying closed loop performance with reduced-order controller

			BT method			BSPA method		
			ρ_1			ρ_1		
			0.5	2	3.5	0.5	2	3.5
5th-order	ρ_2	5	0.185	0.244	0.257	0.175	0.167	0.174
		25	0.252	0.264	0.265	0.170	0.167	0.178
		45	0.271	0.272	0.269	0.166	0.171	0.182
		65	0.279	0.276	0.270	0.165	0.175	0.185
		85	0.283	0.277	0.270	0.169	0.179	0.189
4th-order	ρ_2	5	0.247	0.383	0.457	0.167	0.167	0.176
		25	0.376	0.443	0.524	0.164	0.172	0.181
		45	0.430	0.480	0.524	0.165	0.176	0.185
		65	0.463	0.507	0.545	0.168	0.178	0.188
		85	0.500	0.529	0.567	0.171	0.181	0.191
3th-order	ρ_2	5	1.074	2.444	3.523	0.447	0.450	0.455
		25	2.666	3.695	4.528	0.456	0.459	0.462
		45	3.858	4.660	5.319	0.466	0.486	0.470
		65	4.782	5.421	5.958	0.475	0.477	0.479
		85	5.5120	6.040	6.483	0.484	0.486	0.487

**Figure 2:** Parameter trajectory

varying controller found by BT and BSPA method for all parameter trajectory are described in Figure 3. From these figures, it can be seen that the settling time of the 6th, 5th, and 4th-parameter varying controller found by proposed method are lower than 0.5 seconds and that the closed-loop step responses achieve stability within 0.4 seconds for all parameter trajectory. Note that from Figure 3, the step response of the LPV closed-loop system with full-order LPV controller coincides with that of the LPV closed-loop system with 5th-order controller found by proposed method.

5th-order LTI controllers found by BSPA method. Note that the settling time of step responses of the 3 LTI closed-loop systems with 4th-order controllers found by BSPA method are about 0.55 seconds (see Figure 4). Whereas the settling time of step response of the LPV closed-loop system with 4th-order LPV controller found by proposed method (BSPA generalization) is about 0.35 seconds (see Figure 3).

From above results, the order of the parameter varying controller obtained by BT method and that of time invariant controller found by BSPA method can be reduced only up to 5th. On the other hand the order of the parameter varying controller found by proposed method can be reduced as low as 4th while maintaining the closed-loop system stability and providing the same level of the closed-loop system performance as the full-order parameter varying controller .

6. CONCLUDING REMARKS

We have generalized of the singular perturbation approximation to reduce the order of LPV controller. The BSPA generalization was based on solutions of the parameter varying Lyapunov inequalities of the LPV system. We have shown that the reduced-order LPV controller guaranteed LPV closed-loop stability and performance. The effectiveness of the proposed controller reduction method was verified by applying it to a missile model. The closed-loop performance with the reduced-order LPV controller found by the proposed method and that found by balanced truncation method have been compared. Comparative case studies have confirmed that reduced-parameter varying controller obtained by proposed method achieves a smaller order and generally has a better performance than that obtained by BT method.

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REFERENCES

1. P. APKARIAN AND R.J. ADAM, "Advanced Gain Scheduling Techniques for Uncertain System", *Proceeding American Control Conference* (1997).
2. P. APKARIAN AND J.M. BIANNIE, "Self-Scheduled \mathcal{H}_∞ Control of Missile via Linear Matrix Inequality", *Journal of guidance, control and dynamic* **18** (3) (1995).
3. P. APKARIAN, P. GAHINET AND G. BECKER, "Self-Scheduled H_∞ Control of Linear Parameter Varying Systems : a Design Example", *Automatica* **31** (9) (1995), 1251–1261.
4. B. RIYANTO, "Self-Scheduled H_∞ Control Design for N250 Aircraft Dynamics Based on Polytopic LPV Model", *Proc. of the IASTED International Conference Control and Applications*, Banff, Canada (1999), 52–57.

5. G.S. BECKER, *Quadratic Stability and Performance of Linear Parameter Varying Systems*, Ph.D Dissertation, University of California at Berkeley, 1993.
6. DO CHANG OH, KYEONG HO BANG AND HONG BANG PARK, "Controller order reduction using singular perturbation approximation", *Automatica* **33** (6) (1997), 1203–1207.
7. H.M.H. EL-ZOBAIDI AND I. JAIMOUKHA, "Robust Control and Model and Controller Reduction of Linear Parameter Varying Systems", *Proc. of the 37th IEEE Conference on Decision and Control*, Tampa, Florida, USA **3** (1998), 3015–3020.
8. P. GAHINET, A. NEMIROVSKI, A. LAUB AND M. CHILALI, *The LMI Control Toolbox*, The MathWorks, Inc.: Natick, MA, 1995.
9. P.J. GODDARD, *Performance-Preserving Controller Approximation*, PhD Dissertation, Trinity College Cambridge, 1995.
10. R. SARAGIH AND K. YOSHIDA, "Reduced-Order of Structural Control using Singular Perturbation Approach", *Proc. of the 4th Int. Conf. on MOVIC* **3** (1998), 539–544.
11. WIDOWATI, R. SARAGIH, B. RIYANTO AND S.M. NABABAN, "Reduksi model untuk sistem LPV dengan metode perturbasi singular", *J. Indones. Math. Soc. (MIHMI)* **9** (2) (2003), 151–156.
12. WIDOWATI, R. SARAGIH, B. RIYANTO AND S.M. NABABAN, "Model reduction of linear parameter varying systems", *Proceedings of International Conference on Mathematics and Its Applications*, GMU-Yogyakarta, Indonesia (2003), 376–383.
13. WIDOWATI, B. RIYANTO, R. SARAGIH AND S.M. NABABAN, "Model Reduction for Unstable LPV Systems Based on Coprime Factorizations and Singular Perturbation", *Proceedings of The 5th Asian Control Conference*, Melbourne, Australia (2004), 962–969.
14. WIDOWATI, B. RIYANTO AND R. SARAGIH, "Controller reduction of the parameter dependent systems", *Proceeding ITB on Engineering Science* **36B** (1) (2004), 43–56.
15. G.D. WOOD, P.J. GODDARD AND K. GLOVER, "Approximation of Linear Parameter-Varying Systems", *Proceedings of 35th Conference on Decision and Control* (1996).
16. K. ZHOU, C. D'SOUZA AND J.R. CLOUTIER, "Structurally Balanced Controller Order Reduction with Guaranteed Closed Loop Performance", *System and Control letters* **24** (1995), 235–242.

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