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Model Reduction of LPV Control with Bounded Parameter Variation Rates

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Abstract

This paper proposes a reduced-order controller of linear parameter varying systems. The parameters are available for measurement while their ranges and rates of variation are assumed to be bounded. A balancing parameter varying systems is first presented. Furthermore, a singular perturbation method of linear time invariant systems is generalized to reduce the order of the balanced systems. A reduced-order model can be obtained by setting to zero a derivative all states corresponding to smaller $\mathcal{Q}_e$-singular values. Based on the reduced-order model the low-order parameter varying controllers are designed by using parameter dependent $\mathcal{H}_\infty$ synthesis. Effectiveness of the proposed model reduction is verified by applying it to Moore-Greitzer model of a jet engine compressor.

1 Introduction

Almost all physical systems have parameter dependent representations, but many the current modelling and control of physical systems use Linear Time Invariant (LTI) systems. As operating conditions change, the behavior of the physical systems and the linear time invariant model vary so that the closed loop performance designed by LTI controller may degrade. To cope with this problem, it is required that the parameter dependent is designed using parameter dependent controller. Moreover, modern controller design theories such as $\mathcal{H}_2$ and $\mathcal{H}_\infty$ synthesis usually produce controllers that have the same order as that of the model. Thus, the application of these design techniques to high order models will produce high order controllers. The design and analysis of high-order controllers can cause numerical difficulties while it implementation is very complex. Finding low-order controllers for linear parameter varying system is particularly useful from computation point of view.

A reduced-LPV controller will be found through model reduction. Model reduction of linear time invariant (LTI) systems based on coprime factorization has been published by Li Li and Paganini [1]. They derived linear matrix inequality characterization of expansive and contractive coprime factorizations that maintain structure, and use this to construct a method for structured model reduction. The extension of balanced truncation (BT) to reduce the order of unstable LPV systems by approximating contractive coprime factorizations has been studied by some authors [2, 3]. Balanced singular perturbation approximation (BSPA) of LTI systems have been published by several authors [4, 5]. Further, Widowati, et.al [7] generalized the BSPA method to reduce the model order of unstable LPV systems with *unbounded* parameter variation rates. In previous paper [6], we have derived the reduction error of BSPA method of $\mathcal{Q}_e$-stable system. The upper bound of the reduction error was expressed in term of a $\mathcal{L}_2$-norm bound.

In this paper we propose generalization of the BSPA method to reduce both the $\mathcal{Q}_e$- stable and unstable LPV systems with *bounded* parameter variation rates. In comparison with method proposed in [3], this paper uses BSPA method, whereas in [3] BT method was used to reduce a high order-model. In BT method, the reduced-order model is obtained by truncating balanced states corresponding to smaller $\mathcal{Q}_e$-singular values. In BSPA method, all balanced states are first decomposed into the slow and fast modes by defining the smaller $\mathcal{Q}_e$-singular values as the fast mode, and the rest as the slow mode. Next, a reduced-order model can be obtained by setting the velocity of the fast mode equal to zero. Furthermore, from reduced-order model, the reduced-LPV controller is then designed. There are some techniques to design

LPV controller [8-11]. In this paper we use design technique developed by Lee [8] for constructing LPV controller.

The paper is organized as follows. Section 2 describes induced $\mathcal{L}_2$-norm analysis for LPV system. Results concerning balanced LPV systems are presented in Section 3. In Section 4 we generalize the BSPA method of LTI systems to reduce the order of $\mathcal{Q}_e$- stable LPV systems. Section 5 presents results regarding the approximation of unstable LPV systems based on CRCF reduction. Computational issues for calculating reduced order model is discussed in Section 6. In Section 7 the validity of the proposed method is demonstrated for the Moore-Greitzer compressor. Finally, conclusion is drawn in Section 8.

2 LPV Systems with Bounded Parameter Variation Rates

Before defining parameter-dependent stability for LPV systems using parameter dependent Lyapunov functions, we introduce the concept of the parameter ν-variation set. Let $\mathcal{P} \subset \mathcal{R}^s$ be a compact set and $\{\nu_i\}_{i=1}^s$ are non-negative numbers, then parameter ν-variation set is defined [12] as
$\mathcal{F}_{\mathcal{P}}^{\nu} := \{\rho \in \mathcal{C}^1(\mathcal{R},\ \mathcal{R}^s) : \rho(t) \in \mathcal{P}, |\dot{\rho}_i| \leq \nu_i,$
$i = 1, 2, ..., s,\ \forall t \in \mathcal{R}_+\}$,
where $\nu := [\nu_1 \cdots \nu_s]^T$ and $\mathcal{C}^1$ stands for the class of piecewise continuously differentiable functions. Consider the nth-order LPV systems with bounded parameter variation rates represented by

$$\begin{array}{rcl} \dot{x}(t) & = & A(\rho(t))x(t) + B(\rho(t))u(t), \\ y(t) & = & C(\rho(t))x(t) + D(\rho(t))u(t), \quad \forall \rho \in \mathcal{F}_{\mathcal{P}}^{\nu}, \end{array} \tag{1}$$

where $x(t) \in \mathcal{R}^n$, $y(t) \in \mathcal{R}^p$, and $u(t) \in \mathcal{R}^m$. The state-space matrices (A, B, C, D) are assumed to be continuous function of the parameter vector $\rho \in \mathcal{R}^s$.

Definition 2.1 *Given a state-space representation of an LPV system (1). The causal linear operator* $G_{\mathcal{P}}^{\nu} : \mathcal{L}_{2,e}^+ \rightarrow \mathcal{L}_{2,e}^+,\ u(t) \mapsto y(t)$ *is defined as*

$$y(t) = \int_0^t C(\rho(t))\ \Phi(t,\tau)B(\rho(\tau))u(\tau)d\tau + D(\rho(t))u(t). \tag{2}$$

The LPV systems $G_{\mathcal{P}}^{\nu}$ is $\mathcal{Q}_e$-stable (extended quadratically stable) [3, 12] if there exits a real differentiable positive-definite matrix function $P(\rho(t)) = P^T(\rho(t)) > 0$ such that (for brevity, the dependence of ρ on t is omitted)

$$\sum_{i=1}^{s}(\dot{\rho}_i \frac{\partial P(\rho)}{\partial \rho_i}) + A^T(\rho)P(\rho) + P(\rho)A(\rho) < 0. \tag{3}$$

Lemma 2.1 [3] *Given the LPV system $G(\rho)$, if there exits $W(\rho) = W^T(\rho) > 0$ such that the following two conditions hold,*

i. $\gamma^2 I - D^T(\rho)D(\rho) > 0,$

and

$$ii.\ \begin{bmatrix} E(\rho,\dot{\rho}) & W(\rho)B(\rho) & C^T(\rho) \\ B^T(\rho)W(\rho) & -\gamma I & D^T(\rho) \\ C(\rho) & D(\rho) & -\gamma I \end{bmatrix} < 0, \forall \rho \in \mathcal{F}_{\mathcal{P}}^{\nu}, \tag{4}$$

where $E(\rho,\dot{\rho}) = A^T(\rho)W(\rho) + W(\rho)A(\rho) + \sum_{i=1}^{s}(\dot{\rho}_i \frac{\partial W}{\partial \rho_i})$. *Then the system $G(\rho)$ is $\mathcal{Q}_e$-stable and $\|G(\rho)\|_{i,2} < \gamma$ for any allowable trajectories $\rho \in \mathcal{F}_{\mathcal{P}}^{\nu}$. Furthermore, if the two above conditions are satisfied, then it is said that the LPV system satisfies a $\mathcal{Q}_{e\gamma}$ performance level.*

3 SPA of Balanced LPV systems

Suppose that the LPV system $G_{\mathcal{P}}^{\nu}$ is $\mathcal{Q}_e$-stable. Let $Q(\rho)$ and $P(\rho)$ be observability and controllability Gramians satisfying parameter varying Lyapunov differential inequalities,

$$\sum_{i=1}^{s}(\dot{\rho}_i \frac{\partial Q(\rho)}{\partial \rho_i}) + A^T(\rho)Q(\rho) + Q(\rho)A(\rho) + C^T(\rho)C(\rho) < 0, \tag{5}$$

$$-\sum_{i=1}^{s}(\dot{\rho}_i \frac{\partial P(\rho)}{\partial \rho_i}) + A(\rho)P(\rho) + P(\rho)A^T(\rho) + B(\rho)B^T(\rho) < 0, \forall \rho \in \mathcal{F}_{\mathcal{P}}^{\nu}, \tag{6}$$

where $Q(\rho) = Q^T(\rho) > 0$, $P(\rho) = P^T(\rho) > 0$ are differentiable.
The $\mathcal{Q}_e$-singular values of the LPV system is defined as

$$\sigma_i(\rho) = \sqrt{\lambda_i(Q(\rho)P(\rho))},\ i = 1, 2, 3, \cdots, n. \tag{7}$$

Now, let a continuous differentiable matrix function $T(\rho)$ [3] where $T^{-1}(\rho)$ exists for all $\rho \in \mathcal{F}_{\mathcal{P}}^{\nu}$. Define the balancing state transformation $x(t) = T(\rho)\tilde{x}(t)$. This gives

$$\begin{array}{l} \dot{\tilde{x}}(t) = (T^{-1}(\rho)A(\rho)T(\rho)) - T^{-1}(\rho)\sum_{i=1}^{s}(\dot{\rho}_i \frac{\partial T(\rho)}{\partial \rho_i}))\times \\ \qquad \tilde{x}(t) + T^{-1}(\rho)B(\rho)u(t), \\ y(t) = C(\rho)T(\rho)\tilde{x}(t) + D(\rho)u(t), \forall \rho \in \mathcal{F}_{\mathcal{P}}^{\nu}. \end{array} \tag{8}$$

Then the Gramians are transformed to $\tilde{Q}(\rho) = T^T(\rho)Q(\rho)T(\rho)$ and $\tilde{P}(\rho) = T^{-1}(\rho)P(\rho)T^{-T}(\rho)$ such that

$$\tilde{Q}(\rho) = \tilde{P}(\rho) = \Sigma(\rho), \tag{9}$$

where $\Sigma(\rho)$ is a diagonal matrix which has the $\mathcal{Q}_e$-singular values arranged along its diagonal in descending order $\sigma_1(\rho) \geq \sigma_2(\rho) \geq \cdots \geq \sigma_n(\rho) > 0$, $\rho \in \mathcal{F}_{\mathcal{P}}^{\nu}$. Solutions $Q(\rho)$ and $P(\rho)$ are only required to satisfy inequalities (5) and (6), and are known to be not unique. This indicate that the balanced realizations of LPV systems are not unique. Non-uniqueness of $Q(\rho), P(\rho)$ can be exploited to produce more desirable reduced-order systems[3].

Further, balanced parameter varying realization is written as follows.

$$\begin{aligned} G_{\mathcal{P}}^{\nu} &:= \left[\begin{array}{c|c} H(\rho) & T^{-1}(\rho)B(\rho) \\ \hline C(\rho)T(\rho) & D(\rho) \end{array}\right] \\ &:= \left[\begin{array}{c|c} \bar{A}(\rho,\dot{\rho}) & \bar{B}(\rho) \\ \hline \bar{C}(\rho) & D(\rho) \end{array}\right], \end{aligned} \quad (10)$$

where
$H(\rho) = T^{-1}(\rho)A(\rho)T(\rho) - T^{-1}(\rho)\sum_{i=1}^{s}(\dot{\rho}_i \frac{\partial T(\rho)}{\partial \rho_i})$.
Partition the balanced parameter varying conformably with the Gramian $diag(\Sigma_1(\rho), \Sigma_2(\rho))$ as

$$G_{\mathcal{P}}^{\nu} := \left[\begin{array}{cc|c} \bar{A}_{11}(\rho,\dot{\rho}) & \bar{A}_{12}(\rho,\dot{\rho}) & \bar{B}_1(\rho) \\ \bar{A}_{21}(\rho,\dot{\rho}) & \bar{A}_{22}(\rho,\dot{\rho}) & \bar{B}_2(\rho) \\ \hline \bar{C}_1(\rho) & \bar{C}_2(\rho) & D(\rho) \end{array}\right], \quad (11)$$

with $\bar{A}_{11} \in \mathcal{R}^{r\times r}$, $\bar{A}_{12} \in \mathcal{R}^{r\times(n-r)}$, $\bar{A}_{21} \in \mathcal{R}^{(n-r)\times r}$, $\bar{A}_{22} \in \mathcal{R}^{(n-r)\times(n-r)}$, $\bar{B}_1 \in \mathcal{R}^{r\times m}$, $\bar{B}_2 \in \mathcal{R}^{(n-r)\times m}$, $\bar{C}_1 \in \mathcal{R}^{p\times r}$, $\bar{C}_2 \in \mathcal{R}^{p\times(n-r)}$.

When the system is balanced, states corresponding to smaller $\mathcal{Q}_e$-singular values $(\Sigma_2(\rho))$ represent the fast dynamics of the systems (i.e. its states have very fast transient dynamics and decay rapidly to certain steady value). By using the concept of the SPA method [4], we set to zero the derivative of all states corresponding to $\Sigma_2(\rho)$ to approximate the system (11). This gives reduced-order systems with state-space realization

$$\hat{G}_{\rho_r}^{\nu} := \left[\begin{array}{c|c} A_s(\rho,\dot{\rho}) & B_s(\rho,\dot{\rho}) \\ \hline C_s(\rho,\dot{\rho}) & D_s(\rho,\dot{\rho}) \end{array}\right], \quad (12)$$

where

$$\begin{aligned} A_s(\rho,\dot{\rho}) &= \bar{A}_{11}(\rho,\dot{\rho}) - \bar{A}_{12}(\rho,\dot{\rho})\ (\bar{A}_{22}(\rho,\dot{\rho}))^{-1}\times \\ &\quad \bar{A}_{21}(\rho,\dot{\rho}), \\ B_s(\rho,\dot{\rho}) &= \bar{B}_1(\rho) - \bar{A}_{12}(\rho,\dot{\rho})(\bar{A}_{22}(\rho,\dot{\rho}))^{-1}\bar{B}_2(\rho), \\ C_s(\rho,\dot{\rho}) &= \bar{C}_1(\rho) - \bar{C}_2(\rho)(\bar{A}_{22}(\rho,\dot{\rho}))^{-1}\bar{A}_{21}(\rho,\dot{\rho}), \\ D_s(\rho,\dot{\rho}) &= D(\rho) - \bar{C}_2(\rho)(\bar{A}_{22}(\rho,\dot{\rho}))^{-1}\bar{B}_2(\rho), \end{aligned} \quad (13)$$

assuming that $\bar{A}_{22}(\rho,\dot{\rho})$ is invertible $\forall \rho \in \mathcal{F}_{\mathcal{P}}^{\nu}$.

4 CRCF of LPV Systems

The technique developed in preceding sections is limited to reducing a $\mathcal{Q}_e$-stable LPV system. If the system is not $\mathcal{Q}_e$-stable then the system can not be approximated using the method of previous sections. Hence, we extend of singular perturbation method to reduce the unstable LPV systems by approximating contractive right coprime factorizations (CRCF). In this section, we discuss the CRCF of LPV systems. The characteristics of the CRCF of an LPV systems is defined as follows.

Definition 4.1 [3] *Let $\mathcal{S}_F$ denotes the ring of all causal, $\mathcal{Q}_e$-stable, finite-dimensional LPV systems defined on the underlying feasible parameter set $\mathcal{F}_{\mathcal{P}}^{\nu}$. Denote by $\mathcal{S}_F^-$ the elements in $\mathcal{S}_F$ that have causal inverses. Let $N_{\mathcal{P}}^{\nu} \in \mathcal{S}_F$ and $M_{\mathcal{P}}^{\nu} \in \mathcal{S}_F^-$ have the same number of columns. The ordered pair $[N_{\mathcal{P}}^{\nu},\ M_{\mathcal{P}}^{\nu}]$ represents a contractive right coprime factorization of $G_{\mathcal{P}}^{\nu}$ over $\mathcal{S}_F$ if*

1. *$G_{\mathcal{P}}^{\nu} = N_{\mathcal{P}}^{\nu} M_{\mathcal{P}}^{\nu}$;*
2. *there exist $U_{\mathcal{P}}^{\nu},\ V_{\mathcal{P}}^{\nu} \in \mathcal{S}_F \ni U_{\mathcal{P}}^{\nu} N_{\mathcal{P}}^{\nu} + V_{\mathcal{P}}^{\nu} M_{\mathcal{P}}^{\nu} = I$;*
3. *$[(N_{\mathcal{P}}^{\nu})^T,\ (M_{\mathcal{P}}^{\nu})^T]^T$ is contractive in the following sense*

$$\sup_{\rho \in \mathcal{F}_{\mathcal{P}}^{\nu}} \sup_{\{u \in L_2^+ : \|u\|_2 \leq 1\}} \left\| \begin{bmatrix} N_{\mathcal{P}}^{\nu} \\ M_{\mathcal{P}}^{\nu} \end{bmatrix} u \right\|_{i,2} \leq 1. \quad (14)$$

Now, define the Contractive Right Graph Symbol (CRGS) $\mathcal{G}_{\mathcal{P}}^{\nu} : L_2^+ \mapsto L_2^+ \otimes L_2^+$ of an LPV systems $G_{\mathcal{P}}^{\nu}$, as $\mathcal{G}_{\mathcal{P}}^{\nu} := \begin{bmatrix} N_{\mathcal{P}}^{\nu} \\ M_{\mathcal{P}}^{\nu} \end{bmatrix}$, where $[N_{\mathcal{P}}^{\nu},\ M_{\mathcal{P}}^{\nu}]$ is CRCF of $G_{\mathcal{P}}^{\nu}$. The above definition indicates that $\mathcal{G}_{\mathcal{P}}^{\nu}$ generates the set of all $\mathcal{Q}_e$-stable input-output pairs of the LPV systems $G_{\mathcal{P}}^{\nu}$ by allowing $\mathcal{G}_{\mathcal{P}}^{\nu}$ to act on the whole of L_2^+.

Theorem 4.1 [3] *Let $G_{\mathcal{P}}^{\nu}$ have a continuous, $\mathcal{Q}_e$-stabilizable, and $\mathcal{Q}_e$-detectable realizations. Then CRGS of $G_{\mathcal{P}}^{\nu}$ is given by*

$$\mathcal{G}_{\rho} := \left[\begin{array}{c|c} A(\rho) + B(\rho)F(\rho) & B(\rho)S^{-1/2}(\rho) \\ \hline C(\rho) + D(\rho)F(\rho) & D(\rho)S^{-1/2}(\rho) \\ F(\rho) & S^{-1/2}(\rho) \end{array}\right], \quad (15)$$

where $F(\rho) = -S^{-1}(\rho)(B^T(\rho)X(\rho) + D^T(\rho)C(\rho))$, $S(\rho) = I + D^T(\rho)D(\rho)$, $R(\rho) = I + D(\rho)D^T(\rho)$, and $X(\rho) = X^T(\rho) > 0$ is a solution of the Generalized Control Riccati Inequality (GCRI)

$$\begin{aligned} &\sum_{i=1}^{s}(\dot{\rho}_i \tfrac{\partial X(\rho)}{\partial \rho_i}) + (A(\rho) - B(\rho)S^{-1}(\rho)D^T(\rho)C(\rho))^T \times \\ &X(\rho) + X(\rho)(A(\rho) - B(\rho)S^{-1}(\rho)D^T(\rho)C(\rho)) - \\ &X(\rho)B(\rho)S^{-1}(\rho)B^T(\rho)X(\rho) + C^T(\rho)R^{-1}(\rho)C(\rho) < 0, \\ &\forall \rho \in \mathcal{F}_{\rho}. \end{aligned} \quad (16)$$

Consider Generalized Filtering Riccati Inequality (GFRI)

$$-\sum_{i=1}^{s}(\dot\rho_i \frac{\partial Y(\rho)}{\partial \rho_i}) + (A(\rho) - B(\rho)D^T(\rho)R^{-1}(\rho)C(\rho))\times Y(\rho) + Y(\rho)(A(\rho) - B(\rho)D^T(\rho)R^{-1}(\rho)C(\rho))^T - Y(\rho)C^T(\rho)R^{-1}(\rho)C(\rho)Y(\rho) + B(\rho)S^{-1}(\rho)B^T(\rho) < 0, \forall\rho \in \mathcal{F}_\rho. \tag{17}$$

The connection between the generalized Gramians of CRGS and the solutions of Riccati inequality [3] is given by

$$\bar Q(\rho) = X(\rho), \quad \bar P(\rho) = (I + Y(\rho)X(\rho))^{-1}Y(\rho), \tag{18}$$

where $Q(\rho)$ and $P(\rho)$ are generalized observability and controllability Gramians for $\mathcal{G}^\nu_\mathcal{P}$, respectively. Inequality (16) guarantees that the parameter varying state feedback $F(\rho) = -S^{-1}(\rho)(B^T(\rho)X(\rho) + D^T(\rho)C(\rho))$ will make $\mathcal{G}^\nu_\mathcal{P}$ contractive.

5 CRCF Reduction of LPV Systems with Bounded Parameter Variation Rates

In this section, we propose results regarding the generalization of singular perturbation method for LTI systems to reduce the order of unstable LPV systems with *bounded* parameter variation rates. Consider the CRCF of the nth-order LPV systems $\mathcal{G}_\rho$ in equation (15). By using a balancing state transformation matrix we obtain the transformed controllability and observability Gramians $\tilde P(\rho) = \tilde Q(\rho) = \Sigma(\rho) = diag(\Sigma_1(\rho), \Sigma_2(\rho))$. $\Sigma_1(\rho) = diag(\sigma_1(\rho), \cdots, \sigma_r(\rho))$, $\Sigma_2(\rho) = diag(\sigma_{r+1}(\rho), \cdots, \sigma_n(\rho))$, $\sigma_r(\rho) > \sigma_{r+1}(\rho)$, and $\sigma_j(\rho) = \sqrt{\lambda_j(\bar Q(\rho)\bar P(\rho))}$, $\sigma_j(\rho) \geq \sigma_{j+1}(\rho)$, $j = 1, 2, \cdots, r, r+1, \cdots, n$. A balanced parameter varying CRCF of $\mathcal{G}^\nu_\mathcal{P}$ can be written as follows

$$\mathcal{G}^\nu_\mathcal{P} := \left[\begin{array}{c|c} \bar A(\rho,\dot\rho) + \bar B(\rho)\bar F(\rho) & \bar B(\rho)S^{-1/2}(\rho) \\ \hline \bar C(\rho) + D(\rho)\bar F(\rho) & D(\rho,\dot\rho)S^{-1/2}(\rho) \\ \bar F(\rho) & S^{-1/2}(\rho) \end{array}\right], \tag{19}$$

where
$\bar A(\rho,\dot\rho) = T^{-1}(\rho)A(\rho)T(\rho) - T^{-1}(\rho)\sum_{i=1}^{s}(\dot\rho_i \frac{\partial T}{\partial\rho_i})$,
$\bar B(\rho) = T^{-1}(\rho)B(\rho)$, $\bar C(\rho) = C(\rho)T(\rho)$,
$\bar F(\rho) = -S^{-1}(\rho)(\bar B^T(\rho)\Sigma(\rho) + D^T(\rho)\bar C(\rho))$.

Partition the balanced parameter varying CRCF conformably with $\Sigma(\rho) = diag(\Sigma_1(\rho), \Sigma_2(\rho))$ as follows (the dependence of state space matrices on ρ and $\dot\rho$ is omitted)

$$\mathcal{G}^\nu_\mathcal{P} := \left[\begin{array}{cc|c} \tilde A_{11} + \tilde B_1\tilde F_1 & \tilde A_{12} + \tilde B_1\tilde F_2 & \tilde B_1 S^{-1/2} \\ \tilde A_{21} + \tilde B_2\tilde F_1 & \tilde A_{22} + \tilde B_2\tilde F_2 & \tilde B_2 S^{-1/2} \\ \hline \tilde C_1 + D\tilde F_1 & \tilde C_2 + D\tilde F_2 & DS^{-1/2} \\ \tilde F_1 & \tilde F_2 & S^{-1/2} \end{array}\right], \tag{20}$$

with $\tilde A_{11} \in \mathcal{R}^{r\times r}$, $\tilde A_{12} \in \mathcal{R}^{r\times(n-r)}$, $\tilde A_{21} \in \mathcal{R}^{(n-r)\times r}$, $\tilde A_{22} \in \mathcal{R}^{(n-r)\times(n-r)}$, $\tilde B_1 \in \mathcal{R}^{r\times m}$, $\tilde B_2 \in \mathcal{R}^{(n-r)\times m}$, $\tilde C_1 \in \mathcal{R}^{p\times r}$, $\tilde C_2 \in \mathcal{R}^{p\times(n-r)}$, $\tilde F_1 \in \mathcal{R}^{m\times r}$, $\tilde F_2 \in \mathcal{R}^{m\times(n-r)}$.

Furthermore, the generalized SPA method can be applied to approximate the realization (20) as

$$\mathcal{G}^{\hat\nu}_{\mathcal{P}r} := \begin{bmatrix} N^\nu_\mathcal{P} \\ M^\nu_\mathcal{P} \end{bmatrix} = \left[\begin{array}{c|c} A_s & B_s \\ \hline C_{ns} & D_{ns} \\ C_{ms} & D_{ms} \end{array}\right], \tag{21}$$

where

$$\begin{aligned} A_s &= \tilde A_{11} + \tilde B_1\tilde F_1 - (\tilde A_{12} + \tilde B_1\tilde F_2)(\tilde A_{22} + \tilde B_2\tilde F_2)^{-1}\times (\tilde A_{21} + \tilde B_2\tilde F_1), \\ B_s &= \tilde B_1 S^{-1/2} - (\tilde A_{12} + \tilde B_1\tilde F_2)(\tilde A_{22} + \tilde B_2\tilde F_2)^{-1}\times \tilde B_2 S^{-1/2}, \\ C_{ns} &= \tilde C_1 + D\tilde F_1 - (\tilde C_2 + D\tilde F_2)(\bar A_{22} + \bar B_2\tilde F_2)^{-1}\times (\tilde A_{21} + \tilde B_2\tilde F_1), \\ C_{ms} &= \tilde F_1 - \tilde F_2(\tilde A_{22} + \tilde B_2\tilde F_2)^{-1}(\tilde A_{21} + \tilde B_2\tilde F_1), \\ D_{ns} &= DS^{-1/2} - (\tilde C_2 + D\tilde F_2)(\tilde A_{22} + \tilde B_2\tilde F_2)^{-1}\times \tilde B_2 S^{-1/2}, \\ D_{ms} &= S^{-1/2} - \tilde F_2(\tilde A_{22} + \tilde B_2\tilde F_2)^{-1}\tilde B_2 S^{-1/2}, \end{aligned}$$

assuming that $(\tilde A_{22} + \tilde B_2\tilde F_2)$ is invertible for all $\rho \in \mathcal{F}^\nu_\mathcal{P}$. We obtain the reduced-order model with rth-order, as follows

$$\begin{aligned} \hat G^\nu_{\mathcal{P}r} &:= N^\nu_\mathcal{P}(M^\nu_\mathcal{P})^{-1} \\ &= \left[\begin{array}{c|c} A_s(\rho,\dot\rho) - B_s(\rho,\dot\rho)\mathrm{D}^{-1}_{ms}(\rho,\dot\rho)\mathrm{C}_{ms}(\rho,\dot\rho) & B_s(\rho,\dot\rho)\mathrm{D}^{-1}_{ms}(\rho,\dot\rho) \\ \hline \mathrm{C}_{ns}(\rho,\dot\rho) - \mathrm{D}_{ns}(\rho,\dot\rho)\mathrm{D}^{-1}_{ms}(\rho,\dot\rho)\mathrm{C}_{ms}(\rho,\dot\rho) & D_{ns}(\rho,\dot\rho)\mathrm{D}^{-1}_{ms}(\rho,\dot\rho) \end{array}\right]. \end{aligned} \tag{22}$$

6 Computational Issues

The constrains given by LMIs (5), (6) and (16), (17) are parameter dependent i.e., there is an infinite set of LMIs, one for every value of the parameter. These LMIs can be solved by gridding technique. To be able to solve these infinite set LMIs by gridding, some approximations [8] must be made, by gridding the set $\mathcal{P}$ with a finite L points $\{\rho_k\}^L_{k=1}$. The infinite-dimensional variables $(P(\rho), Q(\rho))$ in LMIs (5), (6) and $(X(\rho), Y(\rho))$ in LMIs (16), (17) are approximated by linear combinations of scalar basis functions. The consequence of this approximation is that the number of LMIs becomes $2L(2^{s+1}+1)$. The number of decision variables becomes $nx(nx+1)(nbasisP + nbasisQ)/2$, where nx is number of states, $nbasisP$ and $nbasisQ$ are number of P and Q basis, respectively. The procedure to compute the reduced-order model of LPV systems is described as follows.

1. Choose sets of continuously differentiable basis functions $\{f_i(\rho)\}^N_{i=1}$ and $\{g_i(\rho)\}^N_{i=1}$.

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2. Parameterize the variables in LMIs (5), (6) for stable LPV systems and (16), (17) for unstable LPV systems, as
   a. $P(\rho) = \sum_{i=1}^{N} f_i(\rho)P_i$, $Q(\rho) = \sum_{i=1}^{N} g_i(\rho)Q_i$.
   b. $X(\rho) = \sum_{i=1}^{N} f_i(\rho)X_i$, $Y(\rho) = \sum_{i=1}^{N} g_i(\rho)Y_i$.

3. Solve for symmetric matrices
   a. $\{P_i\}_{i=1}^{N}$, $\{Q_i\}_{i=1}^{N}$which satisfy the LMIs (5), (6),
   b. $\{X_i\}_{i=1}^{N}$, $\{Y_i\}_{i=1}^{N}$ which satisfy the LMIs (16), (17,
   for all grid points $(\rho, \dot{\rho})$.

4. Repeat 1-3 with more basis functions and/or grid points, if this test fails.

5. Set generalized controllability and observability Gramians
   $\bar{Q}(\rho) = X(\rho), \quad \bar{P}(\rho) = (I + Y(\rho)X(\rho))^{-1}Y(\rho)$.

6. Compute $F(\rho) = -S^{-1}(\rho)(B^T(\rho)X(\rho) + D^T(\rho)C(\rho)), \forall\rho(t) \in \mathcal{F}_{\mathcal{P}}^{\nu}$.

7. Construct a parameter varying CRCF, $G_{\mathcal{P}}^{\nu} = N_{\mathcal{P}}^{\nu}(M_{\mathcal{P}}^{\nu})^{-1}$, with the form (15).

8. Use a similarity transformation to find a balanced realization and partition the balanced realization of the LPV systems corresponding to $\Sigma(\rho) = diag(\Sigma_1(\rho), \Sigma_2(\rho))$.

9. Apply the generalization of the BSPA method to obtain
   a. $\hat{G}_{\mathcal{P}r}^{\nu} := \left[\begin{array}{c|c} A_s(\rho,\dot{\rho}) & B_s(\rho,\dot{\rho}) \\ \hline C_s(\rho,\dot{\rho}) & D_s(\rho,\dot{\rho}) \end{array}\right]$ with state space realization in equation (13).
   b. $\hat{\mathcal{G}}_{\mathcal{P}r}^{\nu} = \begin{bmatrix} N_{\mathcal{P}}^{\nu} \\ M_{\mathcal{P}}^{\nu} \end{bmatrix}$.

10. Form the reduced rth-order model $\hat{G}_{\mathcal{P}_r}^{\nu} = N_{\mathcal{P}}^{\nu}(M_{\mathcal{P}}^{\nu})^{-1}$, $r < n$, with state space realization in equation (22).

Finally, to obtain reduced-order model use procedure at steps 1, $2a$, $3a$, 4, 8, and $9a$ for $\mathcal{Q}_e$-stable LPV systems and steps 1, $2b$, $3b$, $4-8$, $9b$, and 10 for unstable LPV systems. The LMIs here are convex optimization problem which can be solved numerically using LMI Control toolbox for MATLAB [13].

7 Simulation Results

In this section, the developed method in previous section is used to reduce a model of jet engine compressor. This model is taken from Bruzelius [11]. Consider the Moore-Greitzer model of a jet engine compressor,

$$\begin{aligned} &\dot{\Phi} = -\Psi_c - 3\Phi R, \\ &\dot{\Psi} = \tfrac{1}{\beta^2}(\Phi + 1 - v\sqrt{\Psi}), \\ &\dot{R} = \sigma R(1 - \Phi^2 - R), \; R(0) > 0, \end{aligned} \quad (23)$$

where Φ is the annulus averaged mass flow coefficient, Ψ is the plenum pressure rise, R is the squared amplitude of circumferential flow asymmetry, Ψ_c the compressor characteristic relating pressure rise in the plenum to the mass flow, v the control which is proportional to the throttle area and β, σ are system dependent constants.

To be able to describe the compressor model as an LPV system, a state transformation is carried out that moves the non-stall equilibria to the origin [11]. Furthermore, assuming that R and Φ are measured, one LPV systems that represents model (23) is,

$$\dot{x} = \begin{bmatrix} a_{11}(\rho) & 0 & 0 \\ -3\Phi_0 & a_{22}(\rho) & -1 \\ 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 11 \end{bmatrix} u, \quad (24)$$

where $a_{11}(\rho) = \mu(\tilde{\Phi} - \rho_1 - 2\Phi_0\rho_2 - \rho_3)$, $a_{22}(\rho) = \frac{3}{2}\tilde{\Phi}_0 - 3\rho_1 - \frac{3}{2}\Phi_0\rho_2 - \frac{1}{2}\rho_3$, $\rho = \left[r,\ \phi,\ \phi^2\right]^T$, $x = \left[r,\ \phi,\ \psi\right]^T$, $\tilde{\Phi}_0 = 1 - \Phi_0^2$, $\phi = \Phi - \Phi_0, \psi = \Psi - \Psi_c(\phi_0)$, and $r = R$. Taking $R \in [1, 2]$, $\dot{R} \in [-0.5, 0.5]$, $\phi \in [-0.1, 5]$, and $\dot{\phi} \in [-1, 1]$ implies that the parameters are confined in the following set

$\mathcal{F}_{\mathcal{P}}^{\nu} = \{\rho \ :\ 0 \le \rho_1 \le 2,\ -0.1 \le \rho_2 \le 5,\ 0 \le \rho_3 \le 25,\ -0.5 \le \dot{\rho}_1 \le 0.5,\ -1 \le \dot{\rho}_2 \le 1,\ -10 \le \dot{\rho}_3 \le 10\}$.

The control design objectives are to maintain the state variables inside a neighborhood of the equilibrium (the origin) and to keep derivative of the control input u at reasonable level. This can be translated in to setting the penalty to the control signal and its derivative as $W_u = k_1 + \frac{k_2 s}{s+v}$ and to the outputs as $W_T = diag(c_1,\ c_2)$. The augmented plant can be written as,

$$\begin{aligned} \dot{x} &= \begin{bmatrix} b_{11}(\rho) & 0 & 0 & 0 \\ -3.9 & b_{22}(\rho) & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -100 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ -1 \\ -10 \end{bmatrix} u, \\ z &= \begin{bmatrix} 0.001 & 0 & 0 & 0 \\ 0 & 0.1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0.11 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} w, \end{aligned} \quad (25)$$

where $b_{11}(\rho) = (\rho) - 2.76 - 4\rho_1 - 10.4\rho_2 - 4\rho_3$, $b_{22}(\rho) = -1.035 - 3\rho_1 - 1.95\rho_2 - 0.5\rho_3$; $\sigma = 4$ and $\Phi_0 = 1.3$ which corresponds to equilibrium pressure $\Phi_c(1.3) = 1.9984$ or 94% of the peak value $\Phi_c(1) = 2.1496$; and the weight constants $c_1 = 10^{-3}$; $c_2 = 0.1$; $k_1 = 0.001$; $k_2 = 0.1$; and $v = 100$. Next, we calculated reduced-order models by using generalized singular perturbation method suggested in the preceding

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sections. The Procedure outline in §6 is applied to find the reduced-order unstable model (25). For this problem, we pick a gridding of the parameter space $\mathcal{P}$, consisting of 27 points with 3 points in each dimension uniformly (see Table 1).

Table 1: Grid points of the parameter space

	$\rho_2 = -0.1$			$\rho_2 = 2.5$			$\rho_2 = 5$		
$\rho_1 \backslash \rho_3$	0.5	12.5	25	0.5	12.5	25	0.5	12.5	25
0.2	√	√	√	√	√	√	√	√	√
1	√	√	√	√	√	√	√	√	√
2	√	√	√	√	√	√	√	√	√

Basis functions for $X(\rho)$ and $Y(\rho)$ in LMIs (16)-(17) are chosen as follows
$\{f_i(\rho)\}_{i=1}^{27} = \{g_i(\rho)\}_{i=1}^{27} = \{1,\ \rho_1,\ \rho_2,\ \rho_3,\ \rho_1^2,\ \rho_2^2,\ \rho_3^2,\ \rho_1\rho_2,\ \rho_1\rho_3,\ \rho_2\rho_3,\ \rho_1\rho_2^2,\ \rho_1\rho_3^2,\ \rho_1^2\rho_2,\ \rho_3\rho_2^2,\ \rho_1^2\rho_2^2,\ \rho_3^2\rho_2^2,\ \rho_1^2\rho_3^2,\ \rho_1^3,\ \rho_2^3,\ \rho_3^3,\ \rho_1\rho_2^3,\ \rho_1\rho_3^3,\ \rho_1^3\rho_2,\ \rho_2\rho_3^3,\ \rho_1^2\rho_2^3,\ \rho_2^2\rho_3^3,\ \rho_1^2\rho_3^3\}$.
Hence, the parameter dependent $X(\rho)$ and $Y(\rho)$ are in the form of $X(\rho) = \sum_{i=1}^{27} f_i(\rho)X_i$, $Y(\rho) = \sum_{i=1}^{27} f_i(\rho)Y_i$. Using LMI Control toolbox for MATLAB running on pentium(R) 4, 2400 MHz, 18×, 512 MB of RAM we obtain optimal solutions $X(\rho)$ and $Y(\rho)$ after 25 iterations (corresponding to CPU time 5489.1 seconds).

Generalized controllability and observability Gramians are obtained using the solutions $X(\rho)$ and $Y(\rho)$. Then, reduced-order models can be found by applying generalized singular perturbation method to balanced CRCF of the LPV systems (20). Based on the reduced order models the low-order LPV controllers are designed. We use the synthesis procedure developed by Lee [8] to construct LPV controllers. The performance level γ giving the optimal solution of output feedback problems is $\gamma_{opt} = 0.131$ for full-order LPV controller. The evolution of γ during the alternate iterations for calculating the optimal solution of output feedback synthesis of the full-order controller is given in Figure 1. CPU time required to solve output feedback problems is shown in Table 2.

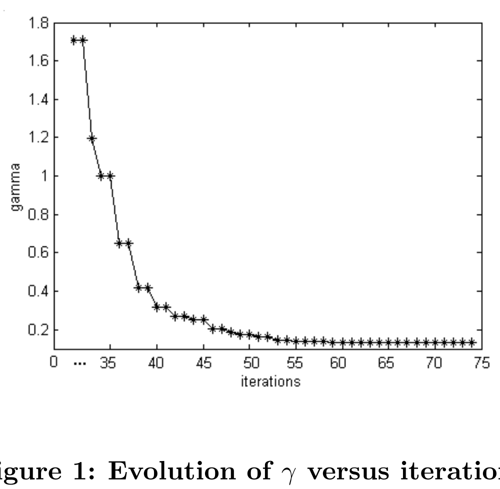

Figure 1: Evolution of γ versus iterations

Table 2: CPU time required for finding controller

Order of the plant	4	3	2	1
CPU time(seconds)	8371.70	3232.61	711.344	193.69

Results presented in Table 2 show that the average time required for finding controller increasing with the plant order. The impulse responses of the closed-loop system with the full-order, 3rd-order, 2nd-order, and 1st-order parameter dependent controllers at grid points of parameter space are depicted in Figure 2.

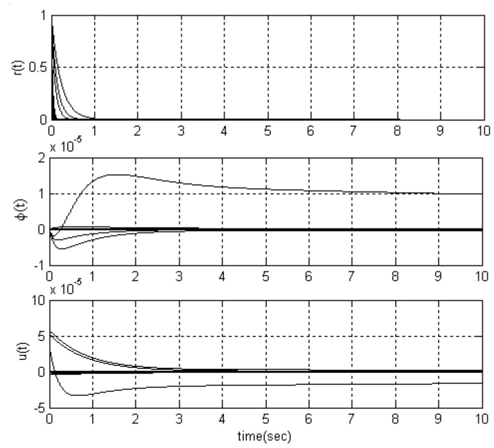

Figure 2a

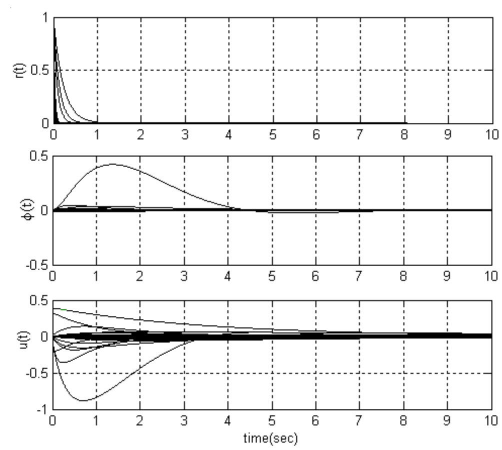

Figure 2b.

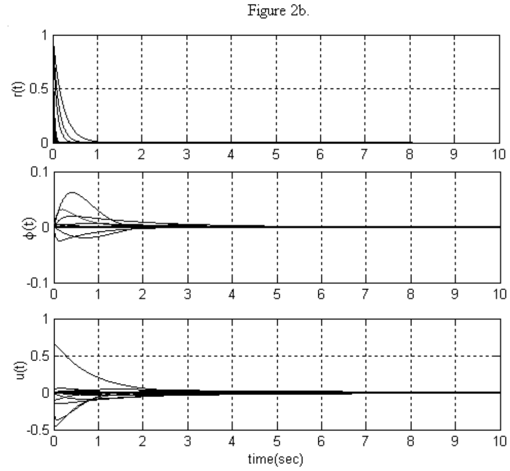

Figure 2c

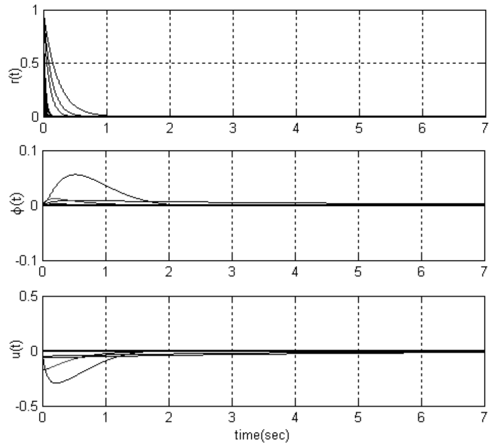

Figure 2: Time responses of closed-loop system with full-order and reduced-order controller found by BSPA method at grid points of parameters (2a: full-order, 2b: 3rd-order, 2c: 2nd-order, 2d: 1st-order controller)

From these figures, it can be seen that the closed-loop systems time responses with full-order and 2nd-, 1st-order controllers are stabilizable within 3 seconds. When the LPV model is reduced to 3th-order, the closed-loop responses reach steady state after much greater than 3 seconds. It may be caused by the difference of 4th- and 3th- $\mathcal{Q}_e$-singular values which is smaller than that of 3th- and 2th-, and that of 2th- and 1th- $\mathcal{Q}_e$singular values, for all grid points of parameter space.

While the numerical study is presented for LPV control of Moore-Greitzer model of jet engine compressor, the result is general enough to be applied to other practical system as well (see for example [6, 7]).

8 Conclusions

We have generalized the balanced singular perturbation approximation (BSPA) of LTI systems to reduce the order of both the $\mathcal{Q}_e$-stable and unstable bounded-rate LPV systems. Complex arising in BSPA generalization of LPV systems is that balanced realizations of the systems are not unique. To reduce conservatism, these can be done by choosing optimal solutions of parameter varying Lyapunov differential inequalities (LDIs) such that the trace of the LDIs solutions product is minimal. The reduced-order model was obtained by setting the derivative of all states corresponding to the smaller $\mathcal{Q}_e$-singular values of the balanced systems equal to zero. The proposed method was shown to be effective in reducing order of the Moore-Greitzer model of jet engine compressor model while maintaining the performance of the closed-loop system with the full-order controller.

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